TWO CLASSES OF PERFECT GRAPHS

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to Mom and Dad
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Abstract

In this work two classes of graphs are introduced. A graph is weakly triangulated if neither the graph nor its complement contain a chordless cycle with five or more vertices as an induced subgraph. A graph is murky if neither the graph nor its complement contain the chordless cycle with five vertices or the chordless path with six vertices as an induced subgraph. The major results of this thesis are theorems concerning these two classes of graphs. In particular, weakly triangulated graphs and murky graphs are perfect.
Résumé

Dans ce travail on présente deux classes de graphes. Un graphe est appelé faiblement triangulé si ni le graphe ni son complément n'admettent de cycle sans corde de cinq sommets ou plus comme sous-graphe induit. Un graphe est appelé troubé si ni le graphe ni son complément n'admettent de cycle sans corde de cinq sommets ou de chemin sans corde de six sommets comme sous-graphe induit. Les résultats les plus importants dans cette thèse sont des théorèmes qui concernent ces deux classes de graphes. En particulier, les graphes faiblement triangulés et les graphes troublés sont des graphes parfaits.
Preface

The thesis consists of four chapters.

Chapter 1 is an overview of the results of the thesis. A perspective of perfect graph theory is presented which motivates the study of weakly triangulated graphs and murky graphs.

Chapter 2 is a brief description of the background of the thesis, namely perfect graph theory. The first section of the chapter is a description of basic definitions and notations of general graph theory. The second section is a brief outline of selected results in perfect graph theory.

Chapter 3 is a collection of results on weakly triangulated graphs. Included are an examination of the relationship between weakly triangulated graphs and star cutsets, and a proof that weakly triangulated graphs are perfect. The chapter also includes algorithms which solve certain optimization problems for weakly triangulated graphs.

Chapter 4 is a collection of results on murky graphs. The highlight of this chapter is a proof that murky graphs are perfect. The proof involves an examination of properties of unbreakable murky graphs; the chapter concludes with a characterization of such graphs.

Unless otherwise stated, the titled theorems in this thesis are the work of the author.
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Chapter 1
Overview

A *clique* is a set of pairwise adjacent vertices in a graph. The *clique number* of a graph is the number of vertices in a largest clique. The *chromatic number* of a graph is the least number of colours needed to colour the vertices, so that adjacent vertices receive different colours. Note that the chromatic number of a graph must be at least as large as the clique number. Claude Berge defined a graph *G* to be *perfect* if, for each induced subgraph *H* of *G*, the chromatic number of *H* is equal to the clique number of *H*.

A graph is *minimal imperfect* if it is not perfect and yet every proper induced subgraph is perfect. It is an easy exercise to check that odd chordless cycles with at least five vertices are minimal imperfect; it is only a little more difficult to show that the complements of such chordless cycles are also minimal imperfect. Are there any other minimal imperfect graphs? The celebrated *Strong Perfect Graph Conjecture*, posed by Berge in 1960, asserts that the answer to this question is "no":

**The SPGC.** A graph is perfect if and only if neither the graph nor its complement contains an odd chordless cycle with five or more vertices.

As early attempts to resolve the *SPGC* were unsuccessful, Berge posed a second conjecture (which, since it is implied by the first, was originally known as the *Weak Perfect Graph Conjecture*):

**The WPGC.** A graph is perfect if and only if its complement is perfect.

The *WPGC* was proved by Lovász (see [1972a] and [1972b]), and is now known as the *Perfect Graph Theorem*. The *SPGC* is still open. The *SPGC* has been the primary
motivation behind most of the research in perfect graph theory to this date.

We call a graph Berge if neither the graph nor its complement contains an odd chordless cycle with five or more vertices. The SPGC asserts that a graph is perfect if and only if it is Berge. This wording of the SPGC suggests one approach to investigating the conjecture: consider particular classes of Berge graphs, and check to see whether or not the graphs in these classes are perfect.

One such class is the class of triangulated graphs, also known as chordal graphs, defined as those graphs in which every cycle with four or more vertices has a chord. Let $C_k$ represent the chordless cycle with $k$ vertices, and $P_k$ the chordless path with $k$ vertices. Let $\overline{G}$ represent the complement of the graph $G$. To see that triangulated graphs are Berge, note that by definition, triangulated graphs do not contain $C_k$ as an induced subgraph, for $k \geq 4$. Also, $C_4$ is an induced subgraph of $\overline{P}_5$, and $\overline{P}_6$ is an induced subgraph of $\overline{C}_j$, for $j \geq 6$; thus triangulated graphs do not contain $\overline{C}_j$ as an induced subgraph, for $j \geq 6$. Finally, since $C_5$ is self-complementary, triangulated graphs do not contain $\overline{C}_5$ as an induced subgraph. To summarize, triangulated graphs do not contain $C_k$, for $k \geq 4$, nor $\overline{C}_j$, for $j \geq 5$, as an induced subgraph. Thus triangulated graphs are Berge.

In 1980 Berge showed that triangulated graphs are perfect; thus triangulated graphs have been known to be perfect almost since the beginning of the history of perfect graph theory. Indeed, Berge’s realization that both triangulated graphs and complements of triangulated graphs are perfect (see Hajnal and Surányi [1958]) was part of the motivation that led him to pose the SPGC and the WPGC.

Another example of a class of Berge graphs is the class of $P_4$-free graphs, defined as those graphs that do not contain $P_4$, the chordless path with four vertices, as an induced
subgraph. Since every $C_j$ contains $P_{j-1}$ as an induced subgraph, $P_4$-free graphs do not contain $C_k$, for $k \geq 5$, as an induced subgraph. Also, since $P_4$ is self-complementary, $P_4$-free graphs do not contain $\overline{C}_k$, for $k \geq 5$, as an induced subgraph. Thus $P_4$-free graphs are Berge. Seinsche [1974] proved that $P_4$-free graphs are perfect.

The main contribution of this thesis is the introduction of two new classes of Berge graphs, together with proofs that such graphs are perfect. In light of the SPGC, it is natural to consider classes of Berge graphs defined in terms of forbidden induced subgraphs, and in terms of chordless cycles and complements of chordless cycles. In light of the Perfect Graph Theorem (formerly the WPGC), it is natural to consider "self-complementary" classes of Berge graphs, i.e. classes of Berge graphs that are closed under complementation. (For example, $C_4$ is not triangulated, whereas $\overline{C}_4$ is; thus the class of triangulated graphs is not self-complementary. On the other hand, if a graph is $P_4$-free, then so is its complement; thus the class of $P_4$-free graphs is self-complementary.) Recall that a graph is triangulated (if and) only if it does not contain $C_k$, for $k \geq 4$, nor $\overline{C}_j$, for $j \geq 5$ as an induced subgraph. The two aforementioned criteria for selecting a "natural" class of Berge graphs suggest the following generalization of triangulated graphs: define a graph to be weakly triangulated if the graph does not contain $C_k$ or $\overline{C}_k$, for $k \geq 5$, as an induced subgraph. Note that the class of weakly triangulated graphs contains all triangulated graphs, all complements of triangulated graphs, and all $P_4$-free graphs.

The second class of Berge graphs introduced in this thesis also contains all $P_4$-free graphs (but not all triangulated graphs). Call a graph murky if it contains no $C_6$, $P_6$ or $\overline{P}_6$ as an induced subgraph. Interest in the class of murky graphs was partly motivated by Hoàng's study of the class of graphs that contain no $C_6$, $P_6$ or $\overline{P}_6$ as an induced
subgraph (see Hoang [1983], and Chvátal, Hoang, Mahadev and De Werra [to appear]).

How can one prove that all graphs in a given class are perfect? One method is to look for some structural property exhibited by all graphs in the class, and then show that no graph with the property can be minimal imperfect. Of particular interest are structural attributes that lead to a decomposition of the graph. For example, suppose that a graph $G$ with vertex set $V$ has a clique cutset, that is, a set of vertices $C$ such that $C$ is a clique, and removal of $C$ leaves a disconnected graph. Let $A$ be any set of vertices that induces a component of $G - C$, and let $B$ be the rest of the vertices of $G$ (i.e. $B = V - A - C$). Then it is a simple exercise to show that $G$ is perfect if the subgraphs induced by $A \cup C$ and $B \cup C$ are perfect: take any two respective minimum colourings of these two graphs, and identify the colours along the clique $C$. Thus a graph with a clique cutset may be decomposed into two smaller graphs, each an induced subgraph of the original graph, in such a way that the original graph is perfect if the two smaller graphs are perfect. This implies that a graph with a clique cutset cannot be minimal imperfect. Dirac [1961] proved that every triangulated graph is either a complete graph or else has a clique cutset. Thus triangulated graphs are perfect.

Another structural property of a graph that leads to a decomposition is a homogeneous set, defined as a subset $H$ of at least two and not all of the vertices of the graph, such that every vertex not in $H$ is adjacent either to all or to none of the vertices of $H$. From a result due to Lovász (see [1972a]) it follows that if a graph $G$ has a homogeneous set $H$, and if $H$ and the graph obtained from $G$ by deleting all but one vertex of $H$ are both perfect, then $G$ is perfect. (Note that both of the smaller graphs are induced subgraphs of the original graph.) Thus a graph with a homogeneous set cannot be minimal imperfect.
Seinsche [1974] proved that every \( P_4 \)-free graph with at least two vertices either is disconnected, or else its complement is disconnected. From this it follows that every \( P_4 \)-free graph with at least three vertices has a homogeneous set. Thus \( P_4 \)-free graphs are perfect. (Although \( P_4 \)-free graphs and homogeneous sets are intimately related, the conclusion that \( P_4 \)-free graphs are perfect can be reached without using homogeneous sets. It is easy to prove that if a graph or its complement is disconnected, then the graph is not minimal imperfect.)

An attribute of a graph that generalizes both a clique cutset and a homogeneous set is a star cutset, defined as a set \( C \) of vertices of a graph \( G \), such that some vertex in \( C \) is adjacent to all remaining vertices in \( C \), and such that \( G - C \) is disconnected. The notion of a star cutset was introduced by Chvátal, with the aim of unifying several structural properties associated with decompositions. Let \( C \) be a star cutset of a graph \( G \), with vertex \( v \) in \( C \) adjacent to all vertices of \( C - v \), and let \( A \) be a component of \( G - C \), and \( B \) the vertices of \( G - C - A \). Chvátal proved that \( G \) is perfect if \( G - v \) and the subgraphs induced by \( A \cup C \) and \( B \cup C \) are perfect; he also proved the analogous decomposition result for the case in which the complement of a graph has a star cutset. It follows that neither a minimal imperfect graph nor its complement can have a star cutset.

As clique cutsets are associated with triangulated graphs, and homogeneous sets with \( P_4 \)-free graphs, one might ask whether there is a class of graphs associated with star cutsets. Since a star cutset is a generalization of both a clique cutset and a homogeneous set (see Chvátal [1985a]), such a class of graphs would include triangulated graphs and \( P_4 \)-free graphs. In fact, there is such a class of graphs, namely weakly triangulated graphs. In Chapter 3 we prove that if a graph is weakly triangulated and has at least
three vertices, then either the graph or its complement has a star cutset. Thus weakly triangulated graphs are perfect. Also, if a graph is not weakly triangulated, then the graph has some induced (not necessarily proper) subgraph (namely, $C_k$ or $\overline{C}_k$ with $k \geq 5$) such that neither the induced subgraph nor its complement has a star cutset; thus star cutsets and weakly triangulated graphs are intimately related.

The star cutset decomposition can be used as the starting point in attempting to prove that other classes of Berge graphs, besides weakly triangulated graphs, are perfect. A graph is called unbreakable if neither the graph nor its complement has a star cutset. Minimal imperfect graphs are unbreakable; thus, in order to show that the graphs of a particular class of Berge graphs are perfect, it suffices to show that the unbreakable graphs of the class are perfect. What do unbreakable Berge graphs look like? What properties do they have? How do chordless cycles (of even length) and complements of such cycles intersect in unbreakable Berge graphs? These questions motivate our proof that murky graphs are perfect; this is the main result of Chapter 4. As a postscript, we include a characterization of unbreakable murky graphs.

One reason perfect graphs are interesting is that there are certain optimization problems which are NP-complete for arbitrary graphs, but for which there exist algorithms which run in polynomial time if the input graph is perfect. A stable set of a graph is a set of pairwise non-adjacent vertices of a graph; the stability number is the number of vertices in a largest stable set. The clique cover number of a graph is the least number of cliques needed to cover the vertices. Note that the stability number of a graph $G$ is equal to the clique number of $\overline{G}$; the clique cover number of $G$ is equal to the chromatic number of $\overline{G}$. Grötschel, Lovász, and Schrijver [1984] described algorithms that solve the problems of determining the clique number, stability number,
chromatic number and clique cover number (and even the weighted versions of these problems) in polynomial time for perfect graphs. Their powerful algorithms are based on the ellipsoid method of linear programming, and on previous work of Lovász [1979] concerning Shannon's capacity of a graph. Given the non-transparent nature of these results, it is of interest to look for simpler algorithms, especially when considering particular classes of perfect graphs. One contribution of this thesis is the presentation of simple combinatorial algorithms which exploit the structure of weakly triangulated graphs to solve the four aforementioned optimization problems (and also the weighted versions of these problems) for the class of weakly triangulated graphs. We have been unable to find analogous algorithms which solve these problems for the class of murky graphs.
Chapter 2

Background

The first section of this chapter is an introduction to the terminology used in the thesis; other definitions will be introduced later as needed. The second section is a brief outline of selected results in perfect graph theory.

2.1 Definitions and Notation

A graph consists of a finite non-empty set of vertices, together with a finite set of edges, or unordered pairs of distinct vertices. If two vertices are in some edge of a graph, then the vertices are said to be adjacent, otherwise they are non-adjacent. We use the terms "sees" and "misses" as synonyms for adjacency and non-adjacency respectively; thus "a sees b and misses c" is equivalent to "a is adjacent to b, but not to c".

A vertex is called a neighbour of another vertex if the two vertices are adjacent. The neighbourhood of a vertex $x$ in a graph $G$, denoted $N(x)$, is the set of all neighbours of $x$ in $G$; the non-neighbourhood of $x$, denoted $\bar{N}(x)$, is the set of all non-neighbours of $x$ in $G - x$.

If $S$ is a subset of the vertices of a graph $G$, then the subgraph of $G$ induced by $S$, denoted $G_S$, is the graph with vertex set $S$, whose edges are precisely those edges of $G$ that consist of two vertices of $S$. An induced subgraph of $G$ is a subgraph induced by some $S$.

A path is a sequence of (pairwise distinct) vertices $v_1v_2 \cdots v_k$, such that every two consecutive vertices $v_j, v_{j+1}$, are adjacent, for $1 \leq j \leq k-1$; if also $v_1$ sees $v_k$, then $v_1v_2 \cdots v_k$ is called a cycle. A chordless path is a path $v_1v_2 \cdots v_k$ such that the only edges of the path are $(v_j, v_{j+1})$, for $1 \leq j \leq k-1$; a chordless cycle is a cycle
$v_1v_2 \cdots v_k$ such that the only edges of the cycle are $(v_j, v_{j+1})$, for $1 \leq j \leq k-1$, and the edge $(v_1, v_k)$. $P_k$ denotes the chordless path with $k$ vertices; $C_k$ denotes the chordless cycle with $k$ vertices.

A graph is connected if for every two vertices $x$ and $y$ there is some path $x \ldots y$. A component of a graph is a maximal connected subgraph. (Throughout the thesis, the terms "maximal" and "minimal" are used with respect to set inclusion; for example, a maximal connected subgraph is a connected subgraph that is not a proper subgraph of any other connected subgraph of the graph). A singleton of a graph is a component with only one vertex; a big component is a component with more than one vertex. A cutset is a set of vertices of a graph, such that the subgraph induced by the remaining vertices is disconnected. Note that in a disconnected graph, any proper subset of the vertices of any component is a subset.

The complement of a graph is the graph obtained by replacing all edges with non-edges, and vice versa. $\overline{G}$ denotes the complement of the graph $G$. Thus, $P_k$ and $C_k$ are the respective complements of $P_k$ and $C_k$.

A clique (respectively stable set) of a graph is a set of pairwise adjacent (respectively non-adjacent) vertices. The clique number (respectively stability number) of a graph is the number of vertices in a largest clique (respectively stable set). The chromatic number (respectively clique covering number) is the minimum number of stable sets (respectively cliques) needed to partition the vertices of a graph. Denote the stability number, clique number, chromatic number and clique covering number of a graph $G$ by $\alpha(G)$, $\omega(G)$, $\chi(G)$ and $\theta(G)$ respectively. A graph is perfect if, for each induced subgraph $H$ of $G$, $\chi(H) = \omega(H)$.
2.2. Some Results in Perfect Graph Theory

In the more than twenty-five years that have passed since Berge posed the SPGC, much research has been directed to the study of perfect graphs. Whereas originally most research was directed towards resolving the conjecture, there are aspects of perfect graph theory which are now considered interesting in their own right, independent of whether or not the SPGC is true (or even if it is resolved). In particular, the emergence in the past two decades of issues related to computational complexity has inspired much interest in perfect graphs: the question of whether or not perfect graphs are in \( NP \) is currently the focus of much research.

In this chapter, we sketch a background of perfect graph theory. A more complete history can be found in any of a number of recently published graph theory texts; for instance, see Berge [1985]. Two books devoted entirely to perfect graph theory are Vol. 83 [1980] and Berge and Chvátal [1984].

2.2.1 The PGT and the SSPGT

When initial attempts to resolve the SPGC were unsuccessful, Berge posed a second conjecture, which (since it was implied by the SPGC) became known as the Weak Perfect Graph Conjecture. This conjecture was proved by Lovász and is now known as the Perfect Graph Theorem.

**PGT** (Lovász [1972a]). A graph is perfect if and only if its complement is perfect.

In light of the PGT, it is natural to look for properties of perfect graphs that are invariant under complementation. Speculation about such properties led Chvátal [1984a] to define the \( P_4 \)-structure of a graph \( G \) as the collection of those sets of four vertices that induce a \( P_4 \) in \( G \). Since the complement of a \( P_4 \) is a \( P_4 \), the \( P_4 \)-structure of
a graph is the same as the $P_4$-structure of its complement. Chvátal conjectured that the perfection of a graph depends only on its $P_4$-structure. This conjecture, implied by the SPGC and implying the PGT(WPGC), was known as the Semi-Strong Perfect Graph Conjecture or SSPGC. The conjecture was proved by Reed in 1984, and is now known as the Semi-Strong Perfect Graph Theorem.

**SSPGT** (Reed [1985]). *Every graph with the $P_4$-structure of a perfect graph is perfect.*

The SSPGC has inspired several results that consider decompositions of perfect graphs defined in terms of $P_4$-structure. For example, vertices $x$ and $y$ are called *siblings* if there is a set $S$ of three vertices such that both $S \cup \{x\}$ and $S \cup \{y\}$ are $P_4$'s. Chvátal proved the following result.

**Theorem** (Chvátal [1985b]). *Let the vertices of a graph $G$ be coloured with two colours such that every two siblings have the same colour. Then $G$ is perfect if and only if each of the subgraphs induced by the set of all vertices of the same colour is perfect.*

This theorem generalizes two earlier results: Chvátal and Hoàng [1985] showed that if the vertices of a graph can be coloured with two colours such that every $P_4$ has an even number of vertices of each colour, then the graph is perfect if and only if each of the two mono-chromatic induced subgraphs is perfect; Hoàng [1985b] showed that if the vertices of a graph can be coloured with two colours in such a way that every $P_4$ has an odd number of vertices of each colour, then the graph is perfect.

Another result concerning $P_4$-structure is that in a minimal imperfect graph every vertex is in at least four $P_4$'s; this follows from a theorem of Olariu [1988]. (Actually, Olariu's theorem is a much stronger statement; however, it is not primarily related to $P_4$-structure.)
2.2.2 Some Classes of Perfect Graphs

From the time that Berge first proposed the SPGC, much of the energy devoted to the study of perfect graphs has focused on finding new classes of perfect graphs. As has been mentioned, both triangulated graphs and complements of triangulated graphs were known to be perfect by 1960. Other classes of graphs long known to be perfect include line graphs of bipartite graphs (this follows from a theorem due to König [1936] concerning the edge-chromatic number of a bipartite graph) and comparability graphs. A graph is a comparability graph if the edges can be directed so that for every three vertices \( a, b, c \), if \((a, b)\) and \((b, c)\) are directed edges, then so is \((a, c)\). It is an exercise to show that comparability graphs are perfect; that complements of comparability graphs are perfect follows from Dilworth's theorem [1950]: the size of a largest anti-chain is equal to the minimum number of chains needed to cover a partially ordered set.

Since the early 1960's many classes of perfect graphs have been discovered. In the rest of this section we briefly discuss two ways of obtaining classes of perfect graphs.

Let \( P \) be some forbidden property of minimal imperfect graphs. If every induced subgraph of a certain graph satisfies \( P \), then the graph is perfect. Thus the "subgraph property" paradigm can be used to define classes of perfect graphs. For example, Berge and Duchet [1984] defined a graph to be strongly perfect if every induced subgraph has a stable set which intersects all maximal cliques. Another class of graphs which fits this paradigm was defined by Meyniel. Call a set \( \{x, y\} \) of vertices of a graph an even pair if every chordless path between \( x \) and \( y \) has an even number of edges. Meyniel [1986] defined a graph \( G \) to be quasi-parity if, for every induced subgraph \( H \) of \( G \) with at least two vertices, either \( H \) or \( \overline{H} \) has an even pair. We will say more about quasi-parity graphs in Chapter 3.
Another way to obtain (candidates for) classes of perfect graphs is to forbid certain
induced subgraphs from Berge graphs. For instance, Tucker [1977] showed that $K_4$-free
Berge graphs are perfect; Parthasarathy and Ravindra showed that $K_{1,3}$-free Berge
graphs [1976] and ($K_4$-e)-free Berge graphs [1979] are perfect. Chvátal and Sbihi refer
to the connected graph with five vertices that consists of a triangle and two pendant
edges as a bull; they showed that bull-free Berge graphs are perfect [1986].

As was mentioned in Chapter 1, the major contribution of this thesis is the
introduction of weakly triangulated graphs and murky graphs. These two classes of
graphs clearly fall into the "forbidden subgraph" paradigm: weakly triangulated graphs
are Berge graphs with no $C_k$ or $\overline{C_k}$, for $k$ even and $k \geq 6$; murky graphs are Berge
graphs with no $P_6$ or $\overline{P_6}$. In fact, weakly triangulated graphs also fall into the
"subgraph property" paradigm; exactly how this is so is discussed in Section 3.2.3.

2.2.3 Properties of Minimal Imperfect Graphs

If the SPGC is true, then the only minimal imperfect graphs are chordless odd
cycles with at least five vertices, and the complements of such cycles. One approach to
the SPGC has been to look for properties of minimal imperfect graphs. For instance, (as
was noted in the previous chapter), a minimal imperfect graph does not have a clique
cutset, nor a homogeneous set, nor a star cutset. (Actually, the fact that a graph does
not have a star cutset implies that is has neither a clique cutset nor a homogeneous set;
see Chvátal [1985a].) A major result in this area is due to Lovász.

**Theorem** (Lovász [1972b]). Every minimal imperfect graph $G$ satisfies
$\alpha(G) \omega(G) = |G| - 1$.

(Recall that $\alpha(G)$ is the stability number of $G$, and $\omega(G)$ the clique number.)
Padberg [1974] extended Lovász's result by showing that in a minimal imperfect graph $G$

- there are $|G|$ largest cliques and $|G|$ largest stable sets.

- every vertex is in exactly $\omega(G)$ largest cliques and $\omega(G)$ largest cliques, and

- every largest stable set intersects all but one largest clique, and vice versa.

Define the graph $C_n^i$ as follows: $v_1, ..., v_n$ are the vertices, with $v_i$ and $v_j$ adjacent

if $|i - j| \leq t$, for every pair of vertices $v_i, v_j$. Observe that $C_{2k+1}^1$ is $C_{2k+1}^1$ and

$C_{2k+1}^1$ is $C_{2k+1}^{k-1}$ . In fact, the graph $C_{\omega+1}^{\omega-1}$ satisfies the conditions of Lovász and

Padberg. Chvátal [1984c] showed the SPGC is equivalent to stating that every minimal imperfect graph has a spanning subgraph isomorphic to $C_{\omega+1}^{\omega-1}$ . However, Chvátal, Graham, Perold and Whitesides [1979] found infinitely many graphs which do not

contain $C_{\omega+1}^{\omega-1}$ as a spanning subgraph, and yet which satisfy the conditions of Lovász

and Padberg; Bland, Huang and Trotter [1979] independently discovered two of these

graphs. Thus the list of properties of minimal imperfect graphs described so far is

insufficient to imply the SPGC.

2.2.4 Complexity and a Changing Perspective

Since the time that the SPGC was first posed, ideas have emerged in the theory of

computer science that have significantly altered the way problems are approached by

computer scientists. One such idea is the notion of a good algorithm, suggested by

Edmonds [1965] as an algorithm which computes the answer to a problem in such a way

that the number of operations required by the algorithm is bounded above by some

polynomial in the size of the problem. This immediately raises the question "for which

problems do there exist good algorithms?".
From this point of view, one of the most important open problems in perfect graph theory is "does there exist a polynomial time algorithm to recognize perfect graphs?" A related question is whether or not there exists a certificate of perfection that could be verified in polynomial time (i.e. whether or not perfect graphs are in \(NP\)). Whitesides has suggested (see Berge and Chvátal [1984], page xii) that perhaps perfect graphs can be created from certain "primitive classes" of perfect graphs using perfection preserving operations. If the graphs in the primitive classes are in \(NP\), and if the perfection preserving operations can be performed in polynomial time, then it would follow that perfect graphs are in \(NP\). For example, \(clique\; identification\) is the process of combining two graphs by identifying a clique of one with a clique of the other. It follows from Dirac's theorem that triangulated graphs can be created from cliques using the perfection preserving operation of clique identification. Whitesides [1984] has shown how to reverse this process, so that every triangulated graph can be decomposed into cliques in polynomial time. (There are faster ways to recognize triangulated graphs; for instance, see Rose, Tarjan and Leuker [1976]. However, the example presented here suffices to illustrate our paradigm.) Although this approach has been successful in showing that certain classes of perfect graphs are in \(NP\) (or even in \(P\)), the question of whether or not perfect graphs are in \(NP\) is still open. On the other hand, \(imperfect\) graphs are in \(NP\). We close the chapter with this result.

Bland, Huang, and Trotter [1979] call a graph \(G\) partitionable if there are integers \(r \geq 2\) and \(s \geq 2\) such that for each vertex \(v\) of \(G\), the vertices of \(G - v\) partition into \(r\) cliques of size \(s\) and \(s\) stable sets of size \(r\). They noted that Lovász's theorem (see the previous section) implies that a graph is minimal imperfect if and only if it contains a partitionable induced subgraph. As Cameron and Edmonds remarked (see Cameron [1982]), this implies that imperfect graphs are in \(NP\).
Chapter 3

Weakly Triangulated Graphs

3.1 Introduction

Recall that a graph is weakly triangulated if it contains no $C_k$, and no $\overline{C}_k$, for $k \geq 5$. In this chapter we describe some properties of weakly triangulated graphs, and show that weakly triangulated graphs are perfect. In particular, we describe a relationship between weakly triangulated graphs and star cutsets. Finally, we describe polynomial time algorithms which solve the maximum clique, maximum independent set, minimum colouring and minimum clique cover problems for weakly triangulated graphs.

An attractive feature of weakly triangulated graphs is that they can be recognized in polynomial time. One such recognition algorithm is as follows: for each vertex in a graph, determine if the vertex is contained in a chordless cycle with five or more vertices; repeat the process for the complement of the graph. Whether or not a vertex $v$ is contained in a chordless cycle with five or more vertices can be checked as follows: for each pair of non-adjacent vertices $x$ and $y$ which are both adjacent to $v$, remove all vertices of the graph adjacent to both $x$ and $y$, as well as all vertices adjacent to $v$ (except $x$ and $y$), and then check whether or not there is a path from $x$ to $y$ in the resulting graph. The vertex $v$ is contained in a chordless cycle with at least five vertices if and only if there exists such a path from $x$ to $y$. For a graph with $n$ vertices and $e$ edges, determining whether or not there is a path between a specified pair of vertices can be done in time $O(e)$. Since the total number of edges in a graph and its complement is $O(n^2)$, the above algorithm recognizes weakly triangulated graphs in time $O(n^5)$. 
Figure 3.1
3.2 Weakly Triangulated Graphs, Star Cutsets, and Perfection

3.2.1 Why Star Cutsets?

In attempting to analyze the structure of weakly triangulated graphs, we begin by examining two special cases: triangulated graphs and $P_4$-free graphs.

Dirac [1961] proved that every minimal cutset in a triangulated graph is a clique. A theorem due to Seinsche [1974] implies that every $P_4$-free graph with at least three vertices has a homogeneous set. However, there are weakly triangulated graphs with no clique cutset, no clique cutset in the complement, and no homogeneous set. The smallest such graph appears in Figure 3.1.

In attempting to unify certain structural properties associated with decompositions of perfect graphs, Chvátal [1985a] conceived the following notion: a star cutset is a set $C$ of vertices of a graph $G$ such that some vertex in $C$ is adjacent to all other vertices in $C$, and such that $G - C$ is disconnected. (In particular, if a graph has a clique cutset, then it has a star cutset; if a graph has a homogeneous set, then either the graph or its complement has a star cutset.) Let $G$ be a graph with star cutset $C$, with vertex $v$ in $C$ adjacent to all vertices of $C - v$, and let $A$ be a component of $G - C$, and $B$ the vertices of $G - C - A$. Chvátal proved that $G$ is perfect if the three subgraphs $G_A \cup C$, $G_B \cup C$, and $G - v$ respectively are perfect; he also proved the analogous decomposition result for the case in which the complement of a graph has a star cutset.

The following is a consequence of these two results:

**The Star Cutset Lemma (Chvátal [1985a]).** If a graph is minimal imperfect, then neither the graph nor its complement has a star cutset.
Chvátal conjectured that every weakly triangulated graph with at least three vertices either has a star cutset, or else its complement has a star cutset. This conjecture will be proved as the WT Star Cutset Theorem.

3.2.2 Perfection

The WT Star Cutset Theorem follows easily from the following theorem.

The WT Min Cut Theorem. Let $N$ be a minimal cutset of a weakly triangulated graph $G$, and let $N$ induce a connected subgraph of $\overline{G}$. Then each connected component of $G - N$ includes at least one vertex adjacent to all the vertices of $N$.

Proof of the WT Min Cut Theorem. We first show that

every two non-adjacent vertices in $N$

have a common neighbour in each component of $G - N$. \hfill (1)

For this purpose, consider arbitrary non-adjacent vertices $x$ and $y$ in $N$, and an arbitrary component $A$ of $G - N$. Since the cutset $N$ is minimal, each vertex in $N$ has at least one neighbour in $A$; now connectedness of $A$ implies the existence of a path from $x$ to $y$ with all interior vertices in $A$; the shortest such path $P$ is chordless. The same argument, applied to another component $B$ of $G - N$, shows the existence of a chordless path $Q$ from $x$ to $y$ with all interior vertices in $B$. The two paths $P$ and $Q$ combine into a chordless cycle in $G$; since $G$ contains no chordless cycle with five or more vertices, each of the two paths must have only one interior vertex. In particular, the interior vertex of $P$ is a common neighbour of $x$ and $y$ in $A$, and (1) is proved.

Next, let us show that

the theorem holds whenever no two vertices in $N$ are adjacent. \hfill (2)

To prove (2), we use induction on $|N|$. When $|N| = 1$, the conclusion follows from the
fact that the cutset $N$ is minimal. When $|N| = 2$, the conclusion is guaranteed by (1). When $|N| \geq 3$, choose distinct vertices $x, y, z$ in $N$ and consider an arbitrary component $A$ of $G - N$. Note that $N - x$ is a minimal cutset of $G - x$, and that $(G - x) - (N - x) = G - N$. Hence the induction hypothesis guarantees the existence of a vertex $u$ in $A$ that is adjacent to all vertices in $N - x$. By the same argument, some vertex $v$ in $A$ is adjacent to all vertices in $N - y$, and some vertex $w$ in $A$ is adjacent to all vertices in $N - z$. We will show that at least one of the vertices $u, v, w$ is adjacent to all the vertices in $N$. Assuming the contrary, note that $u, v, w$ must be distinct. Now $u$ cannot be adjacent to $v$ (else $y, u, v, x$, and any common neighbour of $x$ and $y$ in $G - N - A$, whose existence is guaranteed by (1), would induce a chordless cycle in $G$); by the same argument, $u$ cannot be adjacent to $w$, nor $v$ to $w$. But then $x, w, y, u, z, v$ induce a chordless cycle in $G$. This contradiction completes the proof of (2).

To prove the theorem in its full generality, we again use induction on $|N|$. When $|N| \leq 2$, the conclusion follows from (2). When $|N| \geq 3$, we may assume that at least two vertices in $N$ are adjacent (else the conclusion is guaranteed by (2) again). Now we claim that $N$ includes distinct vertices $x$ and $y$ such that

(i) $x$ and $y$ are adjacent in $G$, and

(ii) both $N - x$ and $N - y$ induce connected subgraphs of $\overline{G}$.

(To justify this claim, we only need choose $x$ and $y$ so that, in the subgraph of $\overline{G}$ induced by $N$, the shortest path from $x$ to $y$ is as long as possible.) Consider an arbitrary component $A$ of $G - N$. By the induction hypothesis, $A$ includes vertices $u$ and $v$ such that $u$ is adjacent to all the vertices in $N - x$ and $v$ is adjacent to all the vertices in $N - y$. We will show that at least one of the vertices $u$ and $v$ is adjacent to all the vertices in $N$. Assuming the contrary, note that $u$ and $v$ must be distinct. By
(i), the shortest path \( P \) from \( z \) to \( y \) in the subgraph of \( \overline{G} \) induced by \( N \) has at least one interior vertex. Now \( u \) and \( v \) must be adjacent: else \( u, v \) and \( P \) would induce a chordless cycle in \( \overline{G} \). Next, the argument showing the existence of \( v \) in \( A \) shows also the existence of a vertex \( r \) in \( G - N - A \) such that \( r \) is adjacent to all the vertices in \( N - y \). If \( r \) is not adjacent to \( y \) then \( u, r, v \) and \( P \) induce a chordless cycle in \( \overline{G} \); else \( u, r, v \) and \( P \) induce a chordless cycle in \( \overline{G} \). This contradiction completes the proof. 

The WT Star Cutset Theorem. If \( G \) is a weakly triangulated graph with at least three vertices then \( G \) or \( \overline{G} \) has a star cutset.

Proof of the WT Star Cutset Theorem. The star cutset may be found as follows. Choose an arbitrary vertex \( w \) in \( G \). For each vertex \( x \) other than \( w \), put \( x \) in the set \( N \) if \( x \) is adjacent to \( w \); else put \( x \) in the set \( M \). If \( N \) is empty then stop: \( \{ u \} \) is a star cutset in \( G \) for every vertex \( u \) in \( M \). If \( M \) is empty then stop: \( \{ v \} \) is a star cutset in \( \overline{G} \) for every vertex \( v \) in \( N \).

Now, both \( M \) and \( N \) are non-empty. If \( M \) induces a disconnected subgraph of \( G \) then stop: \( \{ w \} \cup N \) is a star cutset in \( G \). If \( N \) induces a disconnected subgraph of \( \overline{G} \) then stop: \( \{ w \} \cup M \) is a star cutset in \( \overline{G} \).

Now, \( M \) induces a nonempty connected subgraph of \( G \) and \( N \) induces a nonempty connected subgraph of \( \overline{G} \). If some vertex \( v \) in \( N \) is adjacent to no vertex in \( M \) then stop: \( \{ w \} \cup (N - v) \) is a star cutset in \( G \). In the other case, each vertex in \( N \) is adjacent to at least one vertex in \( M \); note that \( N \) is a minimal cutset in \( G \). Now, the WT Min Cut Theorem guarantees that some vertex \( u \) in \( M \) is adjacent to all the vertices in \( N \). Stop: \( \{ w \} \cup (M - u) \) is a star cutset in \( \overline{G} \).
Corollary. All weakly triangulated graphs are perfect.

Proof. Argue by contradiction; let $G$ be an imperfect weakly triangulated graph. Then there is some induced subgraph $H$ of $G$ such that $H$ is minimal imperfect; $H$ is also weakly triangulated. Graphs with one or two vertices are perfect; thus $H$ has at least three vertices. But now the WT Star Cutset Theorem says that either $H$ or $\overline{H}$ has a star cutset, contradicting Chvátal's Star Cutset Lemma. $lacksquare$

3.2.3 Star Cutsets and Generating Classes of Perfect Graphs

Chvátal has pointed out that a forbidden property of minimal imperfect graphs may be used to generate large classes of perfect graphs from smaller ones. For example, the star cutset may be used in such a way. Specifically, given any class $C$ of graphs, denote by $C^*$ the class of graphs defined recursively by the following two rules:

(i) if $G \in C$ then $G \in C^*$,

(ii) if $G$ or $\overline{G}$ has a star cutset, and if $G - v \in C^*$ for all $v \in G$, then $G \in C^*$.

Chvátal's Star Cutset Lemma implies that $C^*$ is a class of perfect graphs whenever $C$ is. For example, let $\text{Triv}$ denote the class of all graphs with at most two vertices. What can we say about the class of graphs $\text{Triv}^*$? By the WT Star Cutset Theorem it follows that $\text{Triv}^*$ contains the class of weakly triangulated graphs. On the other hand, neither chordless cycles with five or more vertices nor the complements of such cycles have star cutsets; thus a graph in $\text{Triv}^*$ cannot contain $C_k$ or $\overline{C_k}$, for $k \geq 5$, as an induced subgraph. It follows that $\text{Triv}^*$ is exactly the class of weakly triangulated graphs. Thus weakly triangulated graphs are the class of graphs associated with the property "either a graph or its complement has a star cutset".
Another class of graphs associated with star cutsets is the class $Bip^*$, where $Bip$ denotes the class of bipartite graphs. Although $Bip^*$ contains $Triv^*$, as well as many other classes of perfect graphs, it is not known whether or not graphs in $Bip^*$ can be recognized in polynomial time.

3.2.4 Which Weakly Triangulated Graphs Have Star Cutsets?

Note that the WT Star Cutset Theorem states only that a weakly triangulated graph with at least three vertices, or its complement, has a star cutset. We now answer the question "exactly which weakly triangulated graphs have star cutsets?". The following theorem is a strictly stronger statement than the WT Star Cutset Theorem. However, we have included both theorems because the proof of the WT Star Cutset Theorem is much simpler than the proof of the following theorem, and because the WT Star Cutset Theorem suffices to prove that weakly triangulated graphs are perfect. In fact, it is the WT Star Cutset Theorem that appears in Hayward [1985].

The Second WT Star Cutset Theorem. Let $G$ be a weakly triangulated graph. Then exactly one of the following is true:

(i) $G$ is a clique,
(ii) every component of $\overline{G}$ consists of a single edge,
(iii) $G$ has a star cutset.

Before proving the theorem we present a lemma; before presenting the lemma, we introduce some definitions. A vertex $x$ is said to be dominated by a vertex $y$ if every vertex (different from $x$ and $y$) that is adjacent to $x$ is also adjacent to $y$. We call a graph with no dominated vertex domination-free. Recall that $N(x)$ and $M(x)$ are respectively the neighbourhood and non-neighbourhood of a vertex $x$. 
The WT Domination-Free Lemma. If $G$ is a domination-free weakly triangulated graph with at least two vertices, then $\overline{G}$ has a star cutset.

Proof of Lemma. First, we propose to find a vertex $v$ and a component $J$ of $M(v)$ such that

\[\text{every vertex in } N(v) \text{ has a neighbour in } J.\]  

For this purpose, we borrow a trick from Ravindra [1982]: find a vertex $t$ and a component $F$ of $M(t)$ such that the number of vertices in $F$ is minimized (over all choices of $t$ and $F$). We claim that (1) holds whenever $v \in F$ and $J$ is the component of $M(v)$ that contains $t$. To justify this claim, consider an arbitrary $x$ in $N(v)$. We may assume that $x \notin N(t)$, for otherwise $t$ is the neighbour of $x$ in $J$; hence $x \in F$. Since $x$ is not dominated by $v$, it has a neighbour $y$ in $M(v)$; trivially, $y \in F \cup N(t)$. Now we only need verify that $F \cap M(v) \subseteq J$ and $N(t) \cap M(v) \subseteq J$. The second of these inclusions is obvious; to verify the first, we only need verify that every $y$ in $F \cap M(v)$ has a neighbour in $N(t) \cap M(v)$. If the last assertion were false then the component of $M(v)$ that contains $y$ would be contained in $F - v$, contradicting our choice of $t$ and $F$. Hence (1) holds.

Now consider the subgraph $H$ of $G$ induced by $\{v\} \cup N(v) \cup J$. It follows from (1) that $N(v)$ is a minimal cutset in $H$. Next, the WT Min Cut Theorem guarantees that the complement of the subgraph induced by $N(v)$ must be disconnected (otherwise $v$ would be dominated by some vertex of $J$ in $H$, and therefore also in $G$). But then $\{v\} \cup M(v)$ is a star cutset in $\overline{G}$. 

Proof of the Second WT Star Cutset Theorem. We shall argue by induction; the cases where $G$ has at most four vertices can be checked by inspection. Now suppose
that $G$ has at least five vertices. If $G$ is domination-free then $\overline{G}$ is domination-free, and $G$ has a star cutset by the \textit{WT Domination-Free Lemma}. Suppose then that $G$ is not domination-free: in this case there are vertices $u$ and $v$ in $G$, such that $v$ dominates $u$, i.e. $N(u) - v$ is a subset of $N(v)$.

Case 1: suppose that $u$ is not adjacent to all the vertices of $G - v$. In this case $\{v\} \cup N(u)$ is a star cutset.

Case 2: suppose that $u$ is adjacent to all the vertices of $G - v$. Then, since $v$ dominates $u$, $v$ is adjacent to all of $G - u$. There are two subcases to consider.

Case 2.1: suppose that $u$ is adjacent to $v$ (thus $N(v) = G - v$ and $N(u) = G - u$). Then either $G$ is a clique, or else there are non-adjacent vertices $x$ and $y$ in $G$, in which case $G - x - y$ is a star cutset ($v$ is adjacent to all of $G - v$).

Case 2.2: suppose that $u$ is not adjacent to $v$ (thus $N(v) = N(u) = G - u - v$). We now use the inductive hypothesis on $G - u - v$. If $G - u - v$ is a clique, then $G - u - v$ is a star cutset in $G$. If $G - u - v$ has a star cutset $C$, then $C \cup \{u,v\}$ is a star cutset in $G$. Finally, note that the complement of $G$ consists of the complement of $G - u - v$ together with a component consisting of the edge induced by $\{u,v\}$. Thus, if every component of $\overline{G - u - v}$ is a single edge, then every component of $\overline{G}$ is a single edge. This completes the proof of the \textit{Second WT Star Cutset Theorem}.

Vertices $x$ and $y$ of a graph $G$ are called \textit{twins} if every vertex of $G - x - y$ is adjacent either to both $x$ and $y$ or to neither $x$ nor $y$. A corollary of the \textit{Second WT Star Cutset Theorem} is that every twin-free weakly triangulated graph with at least three vertices has a star cutset. This is a stronger statement than the \textit{WT Domination-Free Lemma}.
Figure 3.2
3.2.5 A Domination-Free Weakly Triangulated Graph

Domination-free weakly triangulated graphs are mentioned in the proof of the Second WT Star Cutset Theorem. In this section we describe such a graph $W$. Our search for a domination-free weakly triangulated graph was motivated by Mahadev [1984].

The set of vertices of $W$ is the union of the set $X = \{ x_0, x_1, x_2, \ldots, x_{11} \}$ and the set $Y = \{ y_0, y_1, y_2, \ldots, y_{11} \}$. The only edges of $W$ with both endpoints in $X$ are $(x_{3k}, x_{3k+1})$ and $(x_{3k+1}, x_{3k+2})$, for $k = 0,1,2,3$. The only edges of $\overline{W}$ with both endpoints in $Y$ are $(y_{3k}, y_{3k+1})$ and $(y_{3k+1}, y_{3k+2})$, for $k = 0,1,2,3$. Finally, for $k = 0,1,2,3$, (all indices are modulo 12) the only edge of $W$ between $\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k}, x_{3k+1}, x_{3k+2}\}$ is the edge $(y_{3k}, x_{3k})$.

The only edge of $\overline{W}$ between $\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k+3}, x_{3k+4}, x_{3k+5}\}$ is the edge $(y_{3k}, x_{3k+3})$.

The only edge of $\overline{W}$ between $\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k+6}, x_{3k+7}, x_{3k+8}\}$ is the edge $(y_{3k}, x_{3k+7})$.

The only edge of $W$ between $\{y_{3k}, y_{3k+1}, y_{3k+2}\}$ and $\{x_{3k+9}, x_{3k+10}, x_{3k+11}\}$ is the edge $(y_{3k+11}, x_{3k+9})$.

Table 1 lists that part of the adjacency matrix of $W$ representing edges of the form $(x_i, y_j)$. Figure 3.2 is a drawing of the subgraph of $W$ induced by $X \cup \{y_{3k}, y_{3k+1}, y_{3k+2}\}$, and Figure 3.3 is a drawing of the whole of $W$. Note that $W$ is self-complementary: the permutation $P$ defined by $P(x_i) = y_i$ and $P(y_i) = x_{i+3}$ for $i = 0,1,\ldots,11$ sends edges of $W$ onto edges of $\overline{W}$ and vice versa.
Figure 3.3. The domination-free graph $W$
Table I. \( a_{ij} = 1 \) if and only if \( x_i \) is adjacent to \( y_j \) in \( W \)

Since \( W \) is self-complementary, in order to prove that \( W \) is weakly triangulated it is sufficient to show that \( W \) has no chordless cycle \( C \) with at least 5 vertices. Argue by contradiction: suppose that \( W \) contains such a \( C \). Recall that

(i) the subgraph of \( W \) induced by \( X \) consists of four disjoint \( P_3 \)'s,

(ii) the subgraph of \( W \) induced by \( Y \) consists of four disjoint \( P_3 \)'s.

It is a routine matter to verify the following three claims:

(iii) \( W \) contains no chordless path \((p_1, p_2, p_3, p_4)\) whose intersection with \( X \) is the set \( \{p_2, p_3\} \),

(iv) \( W \) contains no chordless path \((p_1, p_2, p_3, p_4, p_5)\) whose intersection with \( X \) is \( \{p_2, p_3, p_4\} \),

(v) \( W \) contains no chordless cycle \((c_1, c_2, c_3, c_4, c_5)\) whose intersection with \( X \) is \( \{c_2, c_3, c_4\} \).

From (v) and the fact that both \( W \) and \( C_5 \) are self-complementary, it
follows that

(vi) $W$ contains no chordless $(c_1, c_2, c_3, c_4, c_5)$ whose intersection with $X$ is the set $\{c_1, c_2\}$.

Because of (i), $C$ cannot be properly contained in $X$. Because of (ii), $C$ cannot be properly contained in $Y$. Hence, let $C_X$ be the subgraph of $W$ induced by those vertices of $C$ in $X$ and $C_Y$ be the subgraph of $W$ induced by those vertices of $C$ in $Y$. Both $C_X$ and $C_Y$ must consist of disjoint chordless paths. Because of (i), $C_X$ contains no $P_k$ with $k \geq 3$. Because of (iv) and (v), $C_X$ contains no $P_3$. Because of (iii), $C_X$ contains no $P_2$. Thus $C_X$ consists of pairwise non-adjacent vertices. $C_X$ cannot consist of a single vertex, because then $C_Y$ would contain a $P_k$, with $k \geq 4$, contradicting (ii). Thus $C_X$ consists of at least two non-adjacent vertices; hence $C_Y$ consists of (at least two) disjoint chordless paths. But $C_Y$ cannot contain three or more disjoint chordless paths, because then $\overline{C_Y}$ would contain a triangle, contradicting (ii). Thus $C_Y$ consists of exactly two disjoint paths; now (ii) implies that one of these paths is an isolated vertex, and the other has two vertices (each subgraph of $W$ induced by at least four vertices in $Y$ is connected). But then the cycle would have to consist of exactly five vertices $(c_1, c_2, c_3, c_4, c_5)$ whose intersection with $Y$ is $\{c_2, c_4, c_5\}$, contradicting (vi). Thus, $W$ is weakly triangulated.

To verify that $W$ is domination-free, assume the contrary: some vertex $u$ is dominated by a vertex $v$. First, consider the case when $u$ is in $X$. By symmetry, we may assume that $u = x_i$ with $0 \leq i \leq 2$. To see that $v$ cannot be in $Y$, consult Table II.
Table II. Neighbours of \( x_1 \) non-adjacent to \( y_j \) in \( W \).

Thus we must have \( v = x_j \) for some \( j \); considering the subgraph of \( W \) induced by \( X \), we conclude easily that \( 0 \leq j \leq 2 \). But now we only need observe that

- \( y_0 \) sees \( x_0 \) and misses \( x_1, x_2 \).
- \( y_9 \) sees \( x_1, x_2 \) and misses \( x_0 \).
- \( y_6 \) sees \( x_2 \) and misses \( x_1 \).
- \( x_0 \) sees \( x_1 \) and misses \( x_2 \).

Thus \( u \) cannot be in \( X \).

Next, consider the case when \( u \) is in \( Y \). By symmetry, we may assume that \( u = y_i \) with \( 0 \leq i \leq 2 \). To see that \( v \) cannot be in \( X \), observe that \( u \) is adjacent to both \( x_2 \) and \( x_8 \), at least one of which is non-adjacent to \( v \). The only remaining subcase, with \( u \) and \( v \) both in \( Y \), is reduced to a previous subcase by considering the permutation \( P \) that sends \( W \) onto its complement: clearly, \( P(v) \) is dominated by \( P(u) \), and both \( P(u) \) and \( P(v) \) are in \( X \). Thus \( W \) is domination-free.

Incidentally, \( W \) has neither a clique cutset nor a homogeneous set. Furthermore, \( W \) is not strongly perfect. (Recall from Chapter 2 that a graph is strongly perfect if in every induced subgraph there is a stable set that meets all maximal cliques.) In the subgraph of \( W \) induced by \( Z = \{ x_0, x_1, x_2, x_6, x_7, x_8, y_6, y_1, y_6, y_7 \} \) no stable set meets all maximal cliques. To see this, note that the maximal cliques of this graph are
Figure 3.4. The subgraph $W_Z$, with weights on maximal cliques
\{ x_0, x_1, y_1 \}, \{ x_6, x_7, y_1 \}, \{ x_0, y_6, y_0 \}, \{ x_0, y_7 \}, \{ x_6, y_0 \}, \{ x_1, y_2 \}, \{ x_7, z_8, y_1 \}, \\
\{ z_2, y_0 \}, \{ z_8, y_0 \}, \text{ and } \{ y_1, y_7 \}.

Assign to these cliques the integers \(-1, -1, 0, 0, 1, 1, 1, -1, -1, -1, -1\) respectively. The sum of the integers is \(-1\), and yet for each vertex \(v\), the sum of the integers of the cliques that contain \(v\) is \(0\). On the other hand, let \(S\) be a stable set that meets every maximal clique of a graph \(G\). Since a stable set meets a clique in at most one vertex, each maximal clique of \(G\) meets precisely one vertex of \(S\). Thus, if integers are assigned to the maximal cliques of \(G\) such that for each vertex \(v\), the sum of the integers of the cliques that contain \(v\) is \(0\), then the sum of the integers must also be \(0\). Thus \(W_Z\) is not strongly perfect, and so neither is \(W\). A drawing of \(W_Z\), with the maximal cliques labelled as described above, is shown in Figure 3.4.
3.3 Weakly Triangulated Graphs and Two-Pairs

An even pair is a pair of (non-adjacent) vertices in a graph, such that every chordless path between the two vertices has an even number of edges. Meyniel defined a graph $G$ to be strict quasi-parity if every induced subgraph $H$ of $G$ which is not a clique has an even pair. A graph $G$ is quasi-parity if every induced subgraph $H$ of $G$, or its complement $\overline{H}$, is either a clique or has an even pair. Meyniel proved that strict quasi-parity graphs and quasi-parity graphs are perfect. Recently Hoang and Maffray [1986] proved that weakly triangulated graphs are strict quasi-parity. It is not known whether or not strict quasi-parity graphs, or quasi-parity graphs, can be recognized in polynomial time.

Hoang and Maffray showed that weakly triangulated graphs are strict quasi-parity by proving that every weakly triangulated graph which is not a clique has an even pair. In fact, a slightly stronger statement is true. We call a pair of vertices a two-pair if every chordless path which joins the vertices has exactly two edges. The original theorem of Hoang and Maffray was easily modified to yield the following theorem.

The WT Two-Pair Theorem. Every weakly triangulated graph which is not a clique has a two-pair.

Proof. We shall prove a stronger assertion, namely, that all weakly triangulated graphs $G$ other than cliques have the following two properties:

1. If $G$ has no clique cutset then each cutset of $G$ contains a two-pair,

2. $G$ contains a two-pair.

Arguing by induction on the number of vertices, we may assume that both (1) and (2) hold for all weakly triangulated graphs with fewer vertices than $G$. To prove (1) for $G$, consider any minimal cutset $C$ of $G$. By assumption, $C$ is not a clique. We shall
distinguish between two cases.

Case 1. Suppose that \( G_C \) is disconnected. Let \( D \) be the set of vertices of some component of \( G_C \) with at least two vertices (since \( C \) is not a clique, there must be such a set \( D \)). Note that every vertex of \( C - D \) is adjacent to every vertex of \( D \), and that \( D \) is a minimal cutset, not a clique, of \( G - (C - D) \). Thus by inductive assumption, \( D \) contains a two-pair of \( G - (C - D) \); obviously, this two-pair is a two-pair of \( G \).

Case 2. Suppose that \( G_C \) is connected. Let \( B_1, \ldots, B_l \) be the vertex sets of the components of \( G - C \). Now use the WT Min Cut Theorem: in each component \( B_j \), there is some vertex that is adjacent to all of \( C \).

Case 2.1. Suppose that \( |B_j| = 1 \) for all \( j \). Then, by inductive assumption the graph \( G_C \) contains some two-pair \( \{x, y\} \). Clearly \( \{x, y\} \) is a two-pair of \( G \).

Case 2.2. Suppose that \( |B_j| \geq 2 \) for some \( j \). Let \( z \) be any vertex of \( B_j \) that is adjacent to all of \( C \); let \( D \) be the set of vertices of \( G \) that are adjacent to some vertex of \( B_j - z \). Now \( D \) is a minimal cutset of \( G - z \). Note that \( D \) is not empty, and not a clique (otherwise \( D \cup \{z\} \) is a clique cutset of \( G \), contradiction). Thus, by inductive assumption \( D \) contains a two-pair of \( G - z \) which is clearly a two-pair of \( G \).

To prove (2) for \( G \), we may assume that \( G \) has a clique cutset \( C \) (otherwise the desired conclusion follows from (1)). Let \( B_1, B_2, \ldots, B_l \) be the vertex sets of the components of \( G - C \). If some \( G - B_j \) is not a clique then by the induction hypothesis \( G - B_j \) contains a two-pair; since every chordless path in \( G \) with both endpoints in \( G - B_j \) is fully contained in \( G - B_j \), this two-pair is also a two-pair in \( G \). Hence we may assume that each \( G - B_j \) is a clique. This implies that \( t = 2 \) and that \( \{x, y\} \) is a two-pair whenever \( x \in B_1, y \in B_2 \). \( \square \)
A noteworthy distinction between an even pair and a two-pair is that it is easy to check in polynomial time whether or not a pair of vertices is a two-pair: remove the common neighbours, and check whether the original two vertices are in different components of the resulting graph. (We know of no polynomial time algorithm to determine if a pair of vertices is an even pair.) In the next section we build upon this property and derive polynomial time algorithms for solving certain optimization problems for weakly triangulated graphs.
3.4 Optimizing Weakly Triangulated Graphs

3.4.1 Introduction

In this section algorithms are presented which solve the following problems for weakly triangulated graphs in polynomial time.

The Maximum Clique Problem. Find a largest clique in a graph.

The Maximum Stable Set Problem. Find a largest stable set in a graph.

The Minimum Colouring Problem. Find a partition of the vertices into the smallest number of stable sets.

The Minimum Clique Covering Problem. Find a partition of the vertices into the smallest number of cliques.

Algorithms are also presented which solve the weighted versions of these problems. In each of the following problems, assume that a graph $G$ with vertices $v_1, ..., v_n$ and positive integers $w(v_1),..., w(v_n)$ are given. These integers are referred to as weights.

The Maximum Weighted Clique Problem. Find a clique $K$ of $G$, such that the sum of the weights of the vertices of $K$ is maximum, over all cliques of $G$.

The Maximum Weighted Stable Set Problem. Find a stable set $S$ of $G$, such that the sum of the weights of the vertices of $S$ is maximum, over all stable sets of $G$.

The Minimum Weighted Colouring Problem. Find stable sets $S_1, ..., S_i$ and integers $X(S_1),..., X(S_i)$, such that

1. for every vertex $v_j$, the sum of the integers $X(S_i)$ of all sets $S_i$ such that $v_j \in S_i$ is at least $w(v_j)$, and such that

2. the sum of all integers $X(S_1) + ... + X(S_i)$ is minimum, over all sets of integers that satisfy (1).
The Minimum Weighted Clique Covering Problem. Find cliques \( K_1, \ldots, K_i \) and integers \( X(K_1), \ldots, X(K_i) \), such that

1. for every vertex \( v_j \), the sum of the integers \( X(K_i) \) of all sets \( K_i \) such that \( v_j \in K_i \) is at least \( w(v_j) \), and such that

2. the sum of all integers \( X(K_j) + \ldots + X(K_i) \) is minimum, over all sets of integers that satisfy (1).

An algorithm which solves any of the weighted problems can be used to solve the unweighted version of the problem by assigning the weight "1" to all vertices. However, since our algorithms for the unweighted problems are more transparent and more efficient (in the sense of worst time complexity) than the algorithms for the weighted problems, we include both sets of algorithms.

Actually, we present only two algorithms. Algorithm \( OPT \) solves the maximum clique and minimum colouring problem for weakly triangulated graphs; Algorithm \( W-OPT \) solves the weighted versions of these problems. Since the complement of a weakly triangulated graph is weakly triangulated, Algorithms \( OPT \) and \( W-OPT \) can also be used to solve the unweighted and weighted versions respectively of the maximum stable set and minimum clique covering problems: to find a largest stable set of a graph \( G \), find a largest clique of \( \overline{G} \); to find a minimum clique covering of a graph \( G \), find a minimum colouring of \( \overline{G} \).

Our algorithms rely on the fact that every weakly triangulated graph is either a clique or else has a two-pair (see the previous section). The aforementioned optimization problems are easily solved for graphs which are cliques. Given a weakly triangulated graph other than a clique, our algorithms repeatedly find a two-pair, each time transforming the graph in question into a smaller weakly triangulated graph by
"identifying" the two-pair. (We will define this term shortly.) Eventually the original graph is transformed into a clique; the optimization problem is solved for the clique, and the two-pair identification process is reversed, transforming the solution of the optimization problem for the clique to the solution of the optimization problem for the original graph.

3.4.2 The Unweighted Case

Let $G(xy \rightarrow z)$ be the graph obtained by replacing vertices $x$ and $y$ of $G$ with a vertex $z$, such that $z$ sees exactly those vertices of $G - \{x, y\}$ that see at least one of $\{x, y\}$. The identification of $x$ and $y$ and $G$ is the process of replacing $G$ with $G(xy \rightarrow z)$.

In the following algorithm, we specify a colouring by a function $f_G$ that assigns some integer from 1 to $t$ to each vertex, such that adjacent vertices are assigned different integers. Assume that $V(G) = \{v_1, v_2, ..., v_n\}$ is the set of vertices of $G$.

Algorithm OPT($G$).

Input: a weakly triangulated graph $G$.

Output: a largest clique $K_G$ and a minimum colouring $f_G$.

Step 1. Look for a two-pair $\{x, y\}$ of $G$.

If $G$ has no two-pair, then

(a) $K_G \leftarrow V(G)$,

(b) for $i = 1$ to $n$ do $f_G(v_i) \leftarrow i$, and

(c) STOP.

Step 2. $J \leftarrow G(xy \rightarrow z)$.

Step 3. $K_J, f_J \leftarrow$ OPT($J$).
Step 4a. If \( z \notin K_j \) then \( K_j \leftarrow K_G \), else \( (z \in K_j \) and...)  
if \( z \) sees all of \( K_j - \{z\} \) then \( K_G \leftarrow K_j - \{z\} + \{z\}, \)
else \( K_G \leftarrow K_j - \{z\} + \{y\}. \)

Step 4b. \( f_G(x) \leftarrow f_G(y) \leftarrow f_J(z); \)
for each \( v_i \in J - \{x,y\} \) do  
\( f_G(v_i) \leftarrow f_J(v_i). \)  

To prove the correctness of Algorithm OPT, we need to establish several properties concerning the identification of a two-pair in a weakly triangulated graph. One such property is described in the following lemma.

The Identification Lemma. Let \( G \) be a weakly triangulated graph with a two-pair \( \{x,y\} \). Then \( G(xy \rightarrow z) \) is weakly triangulated.

Proof. Let \( H = G(xy \rightarrow z) \). We prove that if \( H \) is not weakly triangulated, then neither is \( G \). Assume that \( H \) is not weakly triangulated. Then there is some subset \( C \) of the vertices of \( H \), such that the subgraph \( H_C \) of \( H \) induced by \( C \) is either \( C_k \) or \( \overline{C_k} \), with \( k \geq 5 \). If \( z \notin C \), then clearly \( G \) is not weakly triangulated. Thus we may assume that \( z \in C \).

Case 1. \( H_C \) is a chordless cycle \( c_1 \ldots c_k \) with \( k \geq 5 \).

Assume without loss of generality that \( z = c_1 \). Then \( c_2 \ldots c_k \) is a chordless path in \( G \). Since \( z \) sees \( c_2 \ldots c_k \), and none of \( c_3 \ldots c_{k-1} \), at least one of \( \{z,y\} \) sees \( c_2 \), and similarly \( c_k \), and neither \( z \) nor \( y \) sees any of \( \{c_3 \ldots c_{k-1}\} \). Now observe that at least one of \( \{x,y\} \) must see both of \( \{c_2,c_k\} \). (Suppose not; assume w.l.o.g. that \( z \) sees \( c_2 \) but not \( c_k \) and that \( y \) sees \( c_k \) but not \( c_2 \). Then \( (z,c_2 \ldots c_k,y) \) is a chordless path with at least six vertices, contradicting the assumption that \( \{x,y\} \) is a two-pair.) Thus assume
w.l.o.g. that \( x \) sees both of \( \{c_2, c_k\} \). Then \( \{x, c_2, ..., c_k\} \) induces a \( C_k \) in \( G \). \( G \) is not weakly triangulated, and the theorem holds in this case.

Case 2. \( \overline{H}_G \) is a chordless cycle \( c_1...c_k \) with \( k \geq 5 \).

Assume without loss of generality that \( z = c_1 \). Thus \( c_2...c_k \) is a \( \overline{P}_{k-1} \) in \( G \), and

(i) \( c_2 \) sees neither \( x \) nor \( y \) and \( c_k \) sees neither \( x \) nor \( y \), and

(ii) every vertex in \( \{c_3, ..., c_k\} \) sees at least one of \( \{x, y\} \).

Now observe that

(iii) \( x \) or \( y \) sees both \( c_3 \) and \( c_4 \).

(\text{Assume the contrary. By (ii) either \( x \) or \( y \) sees \( c_3 \); assume w.l.o.g. that \( x \) sees \( c_3 \).
Since (iii) does not hold, \( x \) does not see \( c_4 \); thus by (ii) \( y \) sees \( c_4 \), and since (iii) does not hold, \( y \) does not see \( c_3 \). But then \( \{x, c_3, c_k, c_4, y\} \) is a \( P_5 \), contradicting the fact that \( \{x, y\} \) is a two-pair in \( G \).}

Assume w.l.o.g. that \( x \) sees both \( c_3 \) and \( c_4 \); let \( m \) be the smallest index greater than four such that \( x \) does not see \( c_m \). Then \( x c_2...c_m \) is a \( \overline{C}_k \), with \( k \geq 5 \), \( G \) is not weakly triangulated, and the theorem holds in this case. \( \blacksquare \)

Another result that will be used in proving the correctness of Algorithm OPT is that two-pair identification does not change the clique size. This follows from a lemma due to Meyniel.

**The Clique Size Lemma** (Meyniel [1986]). *If vertices \( x \) and \( y \) of a graph \( G \) are not joined by any chordless path with three edges, then \( \omega(G(xy \rightarrow z)) = \omega(G) \).*

**The Clique Size Corollary.** *If \( \{x, y\} \) is a two-pair of the weakly triangulated graph \( G \), then \( \omega(G(xy \rightarrow z)) = \omega(G) \).*
The Correctness Theorem. Algorithm OPT finds a largest clique and a minimum colouring of $G$.

Proof. Throughout the proof we let $|f_G|$ and $|f_J|$ denote the number of colours of $f_G$ and $f_J$ respectively. Since the clique size of a graph is never greater than the chromatic number, to prove the theorem it suffices to show that $K_G$ is a clique, that $f_G$ is a colouring, and that $|K_G| = |f_G|$. The proof is by induction on the number of calls of OPT. (Since identification decreases the number of vertices by one, OPT is called at most $n$ times; thus the algorithm terminates.) If OPT is called only once, then the algorithm terminates at Step 1. By the WT Two-Pair Theorem, $K_G = V(G)$ is a clique, $f_G$ is a colouring with $n = |K_G|$ colours, and the theorem holds.

Suppose then that OPT is called more than once; thus the algorithm terminates with Step 4b. Since (by the Identification Lemma) $J$ is weakly triangulated, by the inductive hypothesis we may assume that $K_J$ and $f_J$ are a respectively a clique and a colouring of $J$, such that $|K_J| = |f_J|$. If $z \notin K_J$, then $K_G = K_J$, and $|K_G| = |K_J|$. If $z \in K_J$, then either $x$ or $y$ must see all vertices of $K_J - z$. (Suppose not. Then $x$ misses some $v_i \in K_J$; however, $y$ sees $v_i$, else $z$ would miss $v_i$. Similarly, $y$ misses some $v_j \in K_J$ that sees $z$. But then $xv_j y$ is a chordless path, contradicting the assumption that $\{x, y\}$ is a two-pair.) Thus $|K_G| \geq |K_J|$. Since $K_J$ is a largest clique of $J$, the Identification Lemma implies that $|K_G| = |K_J|$. 

Since no pair of adjacent vertices $a, b$ of $J$ satisfy $f_J(a) = f_J(b)$, no pair of adjacent vertices $a, b$ of $G - \{x, y\}$ satisfy $f_G(a) = f_G(b)$. Finally, let $c$ be a vertex of $G$ that sees at least one of $\{x, y\}$; then $c$ sees $z$ in $J$, and so

$$f_G(c) = f_J(c) \neq f_J(z) = f_G(z) = f_G(y).$$

Thus no pair of adjacent vertices $u, v$ of $G$ satisfy $f_G(u) = f_G(v)$, and $f_G$ is a
colouring. Note that \(|f_G| = |f_f|\). Thus \(|K_G| = |K_f| = |f_f| = |f_G|\), and the theorem is proved. 

A corollary of the Correctness Theorem is that \(\omega(G) = \chi(G)\) if \(G\) is weakly triangulated. Thus (since every induced subgraph of a weakly triangulated graph is weakly triangulated) the Correctness Theorem yields another proof that weakly triangulated graphs are perfect.

We now analyze the complexity of Algorithm \(OPT(G)\). Let \(e\) be the number of edges of \(G\), and \(n\) the number of vertices. Note that a pair of non-adjacent vertices \(x\) and \(y\) in a graph \(G\) is a two-pair if and only if there is no path from \(x\) to \(y\) in \(G - N\), where \(N\) is the set of all vertices of \(G\) that see both \(x\) and \(y\). Determining whether or not two vertices are in the same component of a graph can be done in time \(O(n + e)\). Thus determining whether or not a pair of vertices is a two-pair can be done in time \(O(n + e)\), and Step 1 can be done in time \(O((n + e)n^2)\). Step 2 can be done in time \(O(n)\), as can Steps 4a and 4b. Since Step 3 is executed at most \(n - 1\) times, the worst-case complexity of Algorithm \(OPT\) is \(O((n + e)n^3)\).
Figure 3.5. Quasi-identification
3.4.3 The Weighted Case

In this section we present polynomial time algorithms that solve the weighted versions of the maximum clique, maximum stable set, minimum colouring and minimum clique covering problems for weakly triangulated graphs.

One way to solve the weighted clique problem for a graph $G$ is to replace every vertex $v$ of $G$ with a clique of size $w(v)$, and then solve the unweighted clique problem on the resulting graph. However, this transformation is inefficient if the weights are large. Our solution is more direct.

Define $G(u \rightarrow uv)$ to be the graph obtained from the graph $G$ by replacing the vertex $u$ with vertices $v$ and $w$, such that $v$ sees $w$, and such that $u,v,w$ see exactly the same vertices of $G - u$. This process is referred to as duplication.

We now define an operation that combines identification and duplication. Define $G(xy \rightarrow za)$ to be the graph $H(xb \rightarrow z)$, where $H = G(y \rightarrow ab)$. We refer to the process of replacing $G$ with $G(xy \rightarrow za)$ as quasi-identification.

Quasi-identification is represented in Figure 3.5. Note that $G(xy \rightarrow za)$ is the graph obtained from $G$ by replacing $x,y$ with $z,a$ respectively, such that $z$ sees $a$, $z$ sees exactly those vertices of $G - \{x,y\}$ that see at least one of $\{x,y\}$, and $a$ sees exactly those vertices of $G - \{x,y\}$ that see $y$.

In the following algorithm, the weighted colouring $f_G$ consists of stable sets $S_{G_1}, S_{G_2}, ..., S_{G_t}$ and associated positive integers $X(S_{G_1}), X(S_{G_2}), ..., X(S_{G_t})$.

Algorithm W-OPT($G$).

Input: a weakly triangulated graph $G$.

Output: a max. weighted clique $K_G$ and a min. weighted colouring $f_G$. 
Step 1. Look for a two-pair \( \{x, y\} \) of \( G \).

If \( G \) has no two-pair then

(a) \( K_G \leftarrow \mathcal{V}(G) \).

(b) for \( i \leftarrow 1 \) to \( n \) do

\[
S_{Gi} \leftarrow \{v_i\},
\]

\[
X(S_{Gi}) \leftarrow w(v_i)
\]

(c) STOP.

Step 2. Assume that \( w(x) \leq w(y) \).

If \( w(x) = w(y) \) then

\[
J \leftarrow G(xy \rightarrow z),
\]

\[
w(z) \leftarrow w(x);
\]

else \{ ... thus \( w(x) < w(y) \) ... \}

\[
J \leftarrow G(xy \rightarrow za),
\]

\[
w(z) \leftarrow w(x),
\]

\[
w(a) \leftarrow w(y) - w(x).
\]

Step 3. \( K_J, f_J \leftarrow \text{W-OPT}(J) \).

Step 4a. If \( z \notin K_J \) then \( K_G \leftarrow K_J \), else \( (z \in K_J \) and ...)

if \( y \) sees all of \( K_J - \{a, z\} \) then \( K_G \leftarrow K_J - \{a, z\} + y \)

else \( (...z \) sees all of \( K_J - \{a, z\} \) ...) \( K_G \leftarrow K_J - \{a, z\} + z \).

Step 4b. For each set \( S_{I_i} \) of \( f_J \) do

(i) if \( z \in S_{I_i} \) then \( S_{Gi} \leftarrow S_{I_i} - z + \{x, y\} \), else

\[
\text{if } a \in S_{I_i} \text{ then } S_{Gi} \leftarrow S_{I_i} - a + y, \text{ else}
\]

\[
S_{Gi} \leftarrow S_{I_i},
\]

(ii) \( X(S_{Gi}) \leftarrow X(S_{I_i}) \).
The proof of correctness of Algorithm \textit{W-OPT} parallels the proof of correctness of Algorithm \textit{OPT}. We first show that quasi-identification of a two-pair of a weakly triangulated graph yields a weakly triangulated graph.

\textbf{The Quasi-Identification Lemma.} \textit{Let} \( G \) \textit{be a weakly triangulated graph with a two-pair} \( \{x, y\} \). \textit{Then} \( G(xy \rightarrow za) \) \textit{is weakly triangulated.}\n
\textbf{Proof.} \( G(xy \rightarrow za) = H(xb \rightarrow z) \), where \( H = G(y \rightarrow ab) \). It is easy to check that \( H \) is weakly triangulated and that \( \{z, b\} \) is a two-pair of \( H \). Now the result follows from the \textit{Identification Lemma}. \( \square \)

Next we prove that the process of quasi-identification, together with the reweighting of the new vertices as described in Algorithm \textit{W-OPT}, does not change the weighted clique number of \( G \). Let \( \Omega(G) \) represent the weighted clique number of \( G \) (i.e. the weight of a maximum weighted clique of \( G \)).

\textbf{The Weighted Clique Number Lemma.} \textit{Let} \( G \) \textit{be a weighted weakly triangulated graph with a two-pair} \( \{x, y\} \) \textit{such that} \( w(x) \leq w(y) \). \textit{Let} \( F = G(xy \rightarrow za) \), and \textit{let} \( w(z) = w(x) \) \textit{and} \( w(a) = w(y) - w(x) \). \textit{Then} \( \Omega(G) = \Omega(F) \).

\textbf{Proof.} \( F = G(xy \rightarrow za) = H(xb \rightarrow z) \), where \( H = G(y \rightarrow ab) \). Let \( w(b) = w(x) \); clearly \( \Omega(H) = \Omega(G) \). To prove the lemma we need only show that \( \Omega(F) = \Omega(H) \).

Let \( K_H \) be a clique of \( H \) of maximum weight. Since \( x, b \) are non-adjacent, \( K_H \) contains at most one of these two vertices. If \( K_H \) contains neither \( x \) nor \( b \), then \( K_H \) is a clique of \( F \). If \( K_H \) contains \( x \), then \( K_H - x + z \) is a clique of \( F \) with the same weight as \( K_H \); if \( K_H \) contains \( b \), then \( K_H - b + z \) is a clique of \( F \) with the same weight as \( K_H \). Thus \( \Omega(F) \geq \Omega(H) \).
Now let $K_F$ be a clique of $F$ of maximum weight. If $z \notin K_F$ then $K_F$ is a clique of $H$; if $z \in K_F$ then either $K_F - z + z$ or $K_F - z + b$ is a clique of $H$, and both have the same weight as $K_F$. Thus $\Omega(H) \geq \Omega(F)$. 

The Weighted Correctness Theorem. Algorithm $W$-OPT solves the Maximum Weighted Clique Problem and the Minimum Weighted Colouring Problem for a weakly triangulated graph $G$.

Proof. Let $K_G$ and $f_G$ be as described in Algorithm $W$-OPT. It is easy to check that $K_G$ is a clique, and that $S_{G_i}$ is a stable set, for all $i$. Let $|K_G| = \sum_{v \in K_G} w(v)$ and let $|f_G| = \sum_i X(G_i)$. We wish to show that $f_G$ satisfies property (1) of the definition of the Minimum Weight Colouring Problem, and that $|K_G| = |f_G|$. Note that if $K$ is any clique of a weighted graph, and if $f$ is any colouring that satisfies (1), then $|K| \leq |f|$: thus the equality $|K_G| = |f_G|$ implies that both $K_G$ and $f_G$ are optimal.

We first show that (1) holds for $f_G$. Argue by induction on the number of times Step 1 is executed in $W$-OPT($G$). If Step 1 is executed only once, then $X(S_{G_i}) = w(v_i)$ for all $i = 1, \ldots, n$, and (1) holds.

Suppose then that Step 1 is executed at least twice. Thus the algorithm terminates with Step 4. Assume by induction that (1) holds for the colouring $f_J$ of $J$. Recall that in Step 4b,

the vertex $z$ is replaced (in every set $S_{J_i}$ of $f_J$ that contains $z$) with the pair of vertices $x, y$, and, if $w(x) < w(y)$,

the vertex $a$ is replaced (in every set $S_{J_i}$ of $f_J$ that contains $a$) with the vertex $y$. 

In the case where \( w(x) = w(y) \), we have \( w(z) = w(x) = w(y) \), and so

\[
w(x) = w(z) = \sum_{S_i \supseteq z} X(S_i) = \sum_{S_i \supseteq z} X(S_{G_i}).
\]

\[
w(y) = w(z) = \sum_{S_i \supseteq z} X(S_i) = \sum_{S_i \supseteq z} X(S_{G_i}).
\]

In the case where \( w(x) < w(y) \), we have \( w(x) = w(z) \) and \( w(y) = w(x) + w(z) \), and so

\[
w(x) = w(z) = \sum_{S_i \supseteq z} X(S_i) = \sum_{S_i \supseteq z} X(S_{G_i}).
\]

\[
w(y) = w(z) + w(x) = \sum_{S_i \supseteq z} X(S_i) + \sum_{S_i \supseteq z} X(S_i) = \sum_{S_i \supseteq z} X(S_{G_i}).
\]

Thus property (1) holds for \( f_G \).

Now we wish to show that \( |K_G| = |f_G| \). Argue by induction on the number of executions of Step 1; the result clearly holds if Step 1 is executed exactly once. Assume then that Step 1 is executed more than once; thus the algorithm terminates with Step 4.

By the induction hypothesis, \( |K_f| = |f_f| \).

Now an argument similar to that used in the Correctness Theorem establishes that

\( |K_G| = |K_f| \); thus to finish the proof, we need only show that \( |f_G| = |f_f| \). But this is obviously the case, because there is a one-to-one correspondence between the stable sets of \( f_G \) and \( f_f \), namely \( S_{G_i} \) corresponds to \( S_{f_i} \), and \( X(S_{G_i}) = X(S_{f_i}) \) for all \( i \).

We now analyze the complexity of Algorithm \( W\text{-OPT}(G) \). Let \( e \) be the number of edges of \( G \), and \( n \) the number of vertices. As in Algorithm \( OPT(G) \), Step 1 can be done in time \( O((n + e)n^2) \), and Steps 2, 4a and 4b can be done in time \( O(n^2) \). Now consider Step 3. The graph \( J \) is either \( G(xy \rightarrow z) \) or \( G(xy \rightarrow za) \). In the former case \( J \) has one vertex fewer than \( G \); in the latter case, \( J \) has at least one edge more than \( G \).
sees every vertex of $G - \{z,y\}$ that $x$ sees, $a$ sees every vertex of $G - \{z,y\}$ that $y$
sees, and $z$ sees $a$ whereas $z$ misses $y$). Thus Step 3 is executed at most
$n - 1 + \binom{n}{2} - e$ times, and the worst-case complexity of Algorithm $W$-$OPT$ is
$O((n + e)n^4)$. 
Figure 4.1. \( L_8 \) and \( L_9 \) (bottom and top)
Chapter 4
Murky Graphs

4.1 The Main Result

In this chapter we introduce a new class of Berge graphs, namely murky graphs, and prove that murky graphs are perfect. A graph is *murky* if it contains neither $C_5$, $P_6$, nor $\overline{P}_6$ as an induced subgraph.

Recall (see Chapter 1) that a graph is unbreakable if neither the graph nor its complement has a star cutset. A class $H$ of graphs is called *hereditary* if every induced subgraph of a graph in $H$ is in $H$. Since minimal imperfect graphs are unbreakable, to prove that the graphs in some hereditary class $C$ are perfect, we only need prove that the unbreakable graphs in $C$ are perfect. Clearly murky graphs are hereditary; thus to prove that murky graphs are perfect we need only prove that unbreakable murky graphs are perfect.

The *line graph* $L(G)$ of a graph $G$ is the graph whose vertices correspond to the edges of $G$, such that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ share a vertex. $K_{3,3}$ is the graph with six vertices whose complement consists of two disjoint triangles. $K_{3,3}-e$ is the graph obtained by removing any edge from $K_{3,3}$. We let $L_8$ and $L_9$ denote the line graphs of $K_{3,3}-e$ and $K_{3,3}$ respectively. Drawings of $L_8$ and $L_9$ are shown in Figure 4.1.

There are two kinds of unbreakable murky graphs: those that contain $L_8$ as an induced subgraph, and those that do not. Let $U$ be an unbreakable murky graph. If $U$ contains $L_8$ as an induced subgraph, then $U$ is either $L_8$ or $L_9$. If $U$ does not contain $L_8$ as an induced subgraph, then $U$ can be constructed by taking two copies of a $P_4$-free graph, and adding a specified set of edges between the two copies. The following is a
Figure 4.2. A mirror graph
formal definition of such graphs, which we call "mirror graphs".

Define a mirror partition \([R, S]\) of a graph \(G\) to be a partition of the vertices into
sets \(R = \{r_1, ..., r_t\}\) and \(S = \{s_1, ..., s_t\}\) such that

1. \(G_R\) and \(G_S\) are \(P_4\)-free, and
2. \(r_i\) sees \(r_j\) if and only if \(s_i\) sees \(s_j\) if and only if
   \(r_i\) misses \(s_j\) if and only if \(s_i\) misses \(r_j\), for \(1 \leq i < j \leq t\).

(Note that one consequence of (2) is that \(G_R\) and \(G_S\) are isomorphic.)

Any graph that has a mirror partition is called a mirror graph. With respect to a
mirror partition \([R, S]\) of a mirror graph, a pair of corresponding vertices \(\{r_j, s_j\}\) is a
couple, and \(r_j\) is the mate of \(s_j\) (and vice versa). Note that in a mirror graph the
vertices of a couple may or may not be adjacent. A mirror graph is shown in Figure 4.2.

Recall that vertices \(x\) and \(y\) are twins in a graph \(G\) if every vertex in \(G - \{x, y\}\)
sees both or neither of \(\{x, y\}\). Lovász [1972a] showed that a minimal imperfect graph
does not have twins. Olariu calls vertices \(u\) and \(v\) in a graph \(G\) anti-twins if every
vertex in \(G - \{u, v\}\) sees exactly one of \(\{u, v\}\); he proved that a minimal imperfect
graph does not have anti-twins [1986]. (His proof of this result appears in the appendix.)

Burlet and Uhry (see Lemma 5 in [1984]) observed that every \(P_4\)-free graph with at
least two vertices has twins. (We use this fact in the proof of the following proposition,
and frequently throughout the chapter.) We prove a similar result for mirror graphs.

The Mirror Proposition. Let \(F\) be an induced subgraph of a mirror graph \(G\). If
\(F\) has at least two vertices then \(F\) contains twins or anti-twins.

Proof. Let \([R, S]\) be a mirror partition of \(G\). Define
\[
A = \{i : r_i \in F\}, \quad B = \{j : s_j \in F\}.
\]
If some \( k \) belongs to \( A \cap B \) then \( r_k, s_k \) are anti-twins in \( F \). Hence we may assume that \( A \cap B = \emptyset \). Now let \( F^* \) be the graph induced by all \( r_k \) with \( k \in A \cup B \); let \( r_i, r_j \) be twins in \( F^* \). If \( i \in A, j \in A \) (or \( i \in B, j \in B \)) then \( r_i, r_j \) (or \( s_i, s_j \)) are twins in \( F \); if \( i \in A, j \in B \) (or \( i \in B, j \in A \)) then \( r_i, s_j \) (or \( s_i, r_j \)) are anti-twins in \( F \).  

The main results of this chapter are summarized by the following two theorems. The proof of Theorem 4.1 takes up most of the rest of the chapter. The proof of Theorem 4.2 follows almost immediately from Theorem 4.1, and is presented below.

**Theorem 4.1.** If \( G \) is an unbreakable murky graph, then \( G \) is \( L_8, L_9 \), or a mirror graph.

**Theorem 4.2.** Murky graphs are perfect.

**Proof of Theorem 4.2.** By the Star Cutset Lemma and the fact that murky graphs satisfy the hereditary property, we need only prove that unbreakable murky graphs are perfect; by Theorem 4.1 we need only prove that \( L_8, L_9 \) and mirror graphs are perfect. It is a routine exercise to check that \( L_8 \) and \( L_9 \) are perfect (actually, all line graphs of bipartite graphs are perfect: this follows from a theorem due to König [1936] concerning the edge-chromatic number of a bipartite graph). That mirror graphs are perfect follows from the Mirror Proposition, and the fact that a minimal imperfect graph contains neither twins nor anti-twins.

The proof of Theorem 4.1, which appears at the end of Section 4.3, is preceded by several intermediate results: Sections 4.2 and 4.3 contain lemmas concerning properties of unbreakable mirror graphs. As a postscript, in Section 4.4 we present a theorem which extends Theorem 4.1 to a characterization of unbreakable murky graphs.
Figure 4.3. $L_8$ (top) and its complement (bottom)
4.2 Local Properties of Unbreakable Mirror Graphs

In this section we prove several lemmas concerning unbreakable murky graphs. As almost every result in this section is concerned with graphs which contain or do not contain other graphs as induced subgraphs, the following abbreviation will be adopted: we shall say that a graph contains some other graph if the latter is an induced subgraph of the former. Similarly, a graph properly contains some other graph if the latter is a proper induced subgraph of the former.

The definition of "twins" is extended as follows: given vertices \( x \) and \( y \) and a subset \( H \) of the vertices of \( G \), the vertices \( x \) and \( y \) are called twins with respect to \( H \) if \( x \) and \( y \) see exactly the same set of vertices of \( H \cap (G - \{x,y\}) \). Given a vertex \( v \) and a subset \( X \) of the vertices of a graph, we say that \( v \) is (respectively) null, partial, or universal on \( X \) if \( v \) sees (respectively) none, some but not all, or all, of the vertices of \( X \).

The \( L_8 \) Lemma. If an unbreakable murky graph contains \( L_8 \), then it is either \( L_8 \) or \( L_9 \).

Before proving the lemma, we present two claims. The first states how a vertex can attach to \( L_8 \) in a murky graph; the second is a similar statement, but with the added hypothesis that the graph is unbreakable.

Claim Attach. Let \( X \) be a subset of the vertices of a murky graph \( G \) such that \( X \) induces \( L_8 \), and such that some vertex \( v \) of \( G - X \) is partial on \( X \). Then either there is some vertex \( u \) in \( X \) such that \( u \) and \( v \) are twins with respect to \( X \), or else \( X + v \) induces \( L_9 \).

Proof of Claim. Label the vertices of \( X \) as in Figure 4.3. Let \( u \) be an arbitrary vertex outside \( X \). Consider the following four cases.
Case 1: $v$ misses all of 1,2,3,4.

Since $v$ sees at least one vertex in $X$, assume w.l.o.g. that $v$ sees 5. Now $v$ sees 6 (to avoid a $P_6$ on $v1573$); by rotational symmetry, $v$ seeing 6 forces $v$ to see 7, and $v$ seeing 7 forces $v$ to see 8. But then $X + v$ induces $L_9$.

Case 2: $v$ misses all of 5,6,7,8.

Since $v$ sees at least one vertex in $X$, assume w.l.o.g. that $v$ sees 1. If $v$ sees 3 then $v1573$ is a $C_6$; if $v$ misses 3 then $v15736$ is a $P_6$. Hence this case cannot occur.

Case 3: $v$ sees 1 but misses 2 and 3.

Now $v$ misses 7 (to avoid a $C_6$ on $v1237$).

Subcase 3.1: $v$ sees 6.

Now $v$ misses 5 (to avoid a $C_6$ on $v6375$) and $v$ sees 4 (to avoid a $C_6$ on $v6341$). But then $v$ and 8 are twins with respect to $X$.

Subcase 3.2: $v$ misses 6.

Now $v$ sees 8 (to avoid a $P_6$ on $v18637$) and $v$ sees 5 (to avoid a $P_6$ on $v15736$). But then $v5268$ is a $C_6$. Hence this subcase cannot occur.

Case 4: $v$ sees 6 but misses 5 and 7.

Now $v$ misses at least one of 1,3 (to avoid a $C_6$ on $v1573$) and $v$ misses at least one of 2,4 (to avoid a $C_6$ on $v2574$). But then this case reduces to Case 1 or (possibly rotated) Case 3.

We now show that the proof reduces to one of the previous cases. If $v$ misses all of 1,2,3,4 then it satisfies the hypothesis of Case 1; if $v$ sees all of 1,2,3,4 then it satisfies the hypothesis of Case 2 on $\overline{G_X}$. Hence we may assume that $v$ is partial on $\{1,2,3,4\}$; next, rotational symmetry allows us to assume that $v$ sees 1 and misses 2. If $v$ misses 3
then it satisfies the hypothesis of Case 3; if \( v \) sees 3 then it satisfies the hypothesis of Case 4 on \( \overline{G}_X \). This concludes the proof of \textit{Claim Attach}. \( \blacksquare \)

\textbf{Claim No-Twins.} \textit{Let} \( X \) \textit{be a subset of the vertices of an unbreakable murky graph} \( G \) \textit{such that} \( X \) \textit{induces} \( L_g \). \textit{Then there is no vertex} \( v \) \textit{in} \( G - X \) \textit{such that} \( v \) \textit{is a twin with respect to} \( X \) \textit{of some vertex of} \( X \).

\textbf{Proof of Claim.} Assume the contrary: there is a vertex \( u \) in \( X \) such that the set \( S \) of all twins of \( u \) with respect to \( X \) (including \( u \) itself) has size at least two. Without loss of generality, we may assume that \( u = 1 \) (all other cases reduce to this one by rotation and complementation). Note that \( S \) includes no vertices of \( X \) except 1. Since \( G \) is unbreakable, \( S \) is not a homogeneous set in \( G \). Hence some vertex \( v \) outside \( S \) sees some \( a \) in \( S \) and misses some \( b \) in \( S \); trivially, \( v \notin X \). Let \( A \) and \( B \) denote the subgraphs of \( G \) induced by \( X + a - 1 \) and \( X + a - b \) respectively. Note that \( v \) must be partial on \( X \) (else \( v \) would have precisely one neighbour in \( A \) or precisely seven neighbours in \( B \), contradicting \textit{Claim Attach}) and that \( X + v \) does not induce \( L_g \) (else \( v \) would contradict \textit{Claim Attach} with \( A \) in place of \( X \)). By \textit{Claim Attach}, \( v \) must be a twin with respect to \( X \) of some \( w \) in \( X \); since \( v \notin S \), we have \( w \neq 1 \); now symmetry (swapping 5 with 8, 2 with 4, and 6 with 7) allows us to assume that \( w \) is one of \( 2, 3, 5, 6 \). If \( w = 3 \) or \( w = 6 \) then \( v \) contradicts \textit{Claim Attach} with \( A \) in place of \( X \); if \( w = 2 \) or \( w = 5 \), then \( v \) contradicts \textit{Claim Attach} with \( B \) in place of \( X \). This completes the proof of \textit{Claim No-Twins}. \( \blacksquare \)

\textbf{Proof of the} \( L_g \) \textbf{Lemma.} \textit{Let} \( X \) \textit{be a proper subset of the vertices of an unbreakable murky graph} \( G \) \textit{such that} \( X \) \textit{induces} \( L_g \). Since \( G \) is unbreakable, \( X \) is not a homogeneous set of \( G \), and therefore some vertex \( u \) of \( G - X \) is partial on \( X \).
Figure 4.4. $L_7$
Let $Y = X + \{u\}$. Claim Attach together with Claim No-Twins imply that $Y$ induces $L_9$. Now we need only show that there are no vertices in $G - Y$. Assume the contrary; then there is some vertex $w$ in $G - Y$ that is partial on $Y$. But then it is possible to delete some vertex $v$ of $Y$ so that $w$ sees either at most three or at least five vertices of $Y - \{v\}$. But $Y - v$ induces $L_8$, and since $w$ does not see exactly four vertices of $Y - \{v\}$, $Y + w - v$ does not induce $L_9$. Now either Claim Attach or Claim No-Twins is contradicted.

Let $L$ be the class of murky unbreakable graphs that contain $L_8$ as an induced subgraph and $M$ the class of all other unbreakable murky graphs. From the $L_8$ Lemma it follows that $L$ contains at most two graphs, namely $L_8$ and $L_9$. (We have not yet determined whether $L_8$ and $L_9$ are in $L$. In fact, they are. However, since it is not necessary to establish this in order to prove Theorem 4.1, we postpone this task until Section 4.4.)

We now turn our attention to $M$. By definition, no graph in $M$ contains $L_8$ as an induced subgraph. The following lemma shows that in fact the class $M$ is even more restricted. We define $L_7$ to be the graph obtained by removing any vertex of degree four from $L_8$.

**The $L_7$ Lemma.** No graph in $M$ contains $L_7$.

**Proof.** Let $G$ be a graph in $M$. Argue by contradiction; suppose that $X$ is a set of vertices such that $G_X$ is $L_7$, labelled as in Figure 4.4. (The graph in Figure 4.4 can be obtained from the graph in Figure 4.3 by removing vertex 1.) Since $G$ is unbreakable, there must be some path from 5 to 8, none of whose vertices is 3 or sees 3. Consider any shortest such path $P$. Since $G$ is murky, $P$ contains at most three
interior vertices.

Claim 1: $P$ does not contain exactly one interior vertex.

Suppose it did; label the interior vertex 1, so that $P = 518$. Note that 1 misses at least one of 2, 4, 6, 7 (to avoid a $P_6$ on 137245); assume without loss of generality that 1 misses 7. Now 1 sees 4 (to avoid a $C_5$ on 15748), and 1 misses 6 (to avoid a $C_5$ on 16375), and so 1 sees 2 (to avoid a $C_5$ on 18625). But then $\{1,\ldots,8\}$ induces $L_8$, contradiction.

Claim 2: $P$ does not contain exactly two interior vertices.

Suppose it did; label the vertices 0 and 1 so that $P = 5018$. Then 0 sees 7 (suppose not: then (if 0 sees 6) 05736 is a $C_5$ or (if 0 misses 6) 057388 is a $P_6$). By symmetry, 0 sees 2, 1 sees 4, and 1 sees 6. Now, 0 misses 4 (to avoid a $P_6$ on 035427). By symmetry, 0 misses 6, 1 misses 2, and 1 misses 7. But then 02314 is a $C_5$, contradiction.

Claim 3: $P$ does not contain exactly three interior vertices.

Suppose it did; label the vertices 0, 0, 1 so that $P = 50018$. Arguing as in Claim 2, vertex 0 sees 7 and 2 but misses 4 and 6, vertex 1 sees 4 and 6 but misses 7 and 2. Now the graph induced by $\{9,7,4,1,6,2,3\}$ is isomorphic to that induced by $\{2,\ldots,8\}$; furthermore, 0 sees 9 and 1 but misses 3. Therefore, by Claim 1, $\{9,7,4,1,6,2,3,0\}$ induces $L_8$, contradiction. $\blacksquare$
The next lemma is of the following form: if a graph $G$ in $M$ properly contains a certain subgraph $S$, then a certain subgraph $T$ of $G$ properly contains $S$. In this case, $S = P_5$ and $T = C_6$. Later, we present another lemma of this form.

**The P$_5$ Lemma.** Let $G$ be a graph in $M$. Then every $P_5$ in $G$ is contained in a $C_6$.

**Proof.** We will call a $P_5$ bad if it is not contained in a $C_6$. We begin with a simple observation.

*If $abcde$ is a bad $P_5$ in $G$, and some vertex $f$ sees $a$ but not $c$, then $f$ sees $b$. (\*) (Otherwise, $abcd$ is a $C_5$ or $fabcde$ is a $P_5$.)*

Define a **bypass** of a $P_5$ $abcde$ to be a chordless path $P$ from $a$ to $e$, such that every interior vertex of $P$ misses $c$. Note that in an unbreakable graph, every $P_5$ $abcde$ has a bypass (otherwise, $c$ is in some star cutset that separates $a$ and $e$); we will use this fact repeatedly in the proof. Define the **index** of a $P_5$ (in an unbreakable graph) to be the number of interior vertices in a shortest bypass. Note that in a murky graph, the index of a $P_5$ is at most three.

Let $G$ be a graph in $M$. To prove the lemma, we will show that there is no bad $P_5$ in $G$; we do this by showing that there is no bad $P_5$ with index one, two, or three.

**Claim 1:** No bad $P_5$ has index one.

Assume the contrary; let $12345$ be a bad $P_5$, with bypass $P = 185$. By (*), $5$ sees $2$ and $4$. The graph induced by $\{1,\ldots,6\}$ is shown in Figure 4.5.1. Now, $63142$ is a $P_5$ in $\overline{G}$; furthermore, it is a bad $P_5$ of $\overline{G}$. (Assume the contrary; then there is a vertex $7$ that sees $3,1,4$ but misses $2,6$ in $G$. If $7$ sees $5$ then $73265$ is a $C_5$, else $\{1,\ldots,7\}$ induces $L_7$; contradiction.) Now $63142$ must have a bypass in $\overline{G}$. 
Claim 1.1: 63142 does not have index one.

Assume the contrary; let $Q = 672$ be a bypass of 63142 in $\overline{G}$. Thus, using (*) with 63142, in $G$, 7 sees 1 but misses 6, 3, 4, 2. But 7 seeing 1 and missing 2, 3 contradicts (*) with 12345. This concludes Claim 1.1.

Claim 1.2: 63142 does not have index two.

Assume the contrary; let $Q = 6782$ be a bypass of 63142 in $\overline{G}$. Thus, in $G$, vertex 7 sees 2, 1, but misses 6, 8; vertex 8 sees 6, 1 but misses 2, 7. Using (*) with 63142, 7 misses 3, and 8 misses 4; using (*) with 12345, 8 sees 3. Now it follows that

7 misses 4 (to avoid a $C_5$ on 74381),

7 misses 5 (to avoid a $C_5$ on 72345),

8 sees 5 (to avoid a $P_6$ on 718345).

The subgraph of $G$ induced by \{1, 7, 8\} is now the graph in Figure 4.5.2. Now note that 71643 is a bad $P_6$. (Assume the contrary: let 716439 be a $C_6$. Then 9 sees 7, 3 but misses 1, 4, 6. Thus 9 misses 5 (to avoid a $C_5$ on 97165) and 9 sees 8 (to avoid a $C_5$ on 97183); finally, if 9 misses 2 then 97268 is a $C_5$, if 9 sees 2 then 913782 is a $P_6$.)

Claim 1.2.1: 71643 does not have index one.

Assume the contrary; let $R = 793$ be a bypass of 71643. Thus, using (*) with 71643, vertex 9 sees 7, 1, 4, 3 but misses 6. But if 9 misses 2 then 97264 is a $C_5$, if 9 sees 2 then 963142 is a $P_6$. This concludes Claim 1.2.1.

Claim 1.2.2: 71643 does not have index two.

Assume the contrary; let $R = 7903$ be a bypass of 71643. By (*) with 71643, vertex 9 sees 7, 1, 4, 3, but misses 6; vertex 0 sees 3, 4, 9 but misses 7, 6. Now

0 misses 1 (if 0 sees 1 then 0 misses 2 (to avoid a $P_6$ on 063142), and so 05623 is a $C_5$ or \{1, 2, 3, 4, 5, 6\} induces $L_7$.)
9 sees 4  (to avoid a $C_5$ on 91640),
9 sees 8  (to avoid a $C_5$ on 91834),
0 misses 8 (to avoid a $P_6$ on 960148),
9 sees 2  (to avoid a $C_5$ on 91234),
0 misses 2 (to avoid a $P_6$ on 960142), and finally

if 0 sees 5, then 05623 is a $C_5$, else 045812 is a $P_6$. This concludes Claim 1.2.2.

Claim 1.2.3: 71643 does not have index three.

Assume the contrary; let $R = 70x03$ be a bypass of 71643. By (*) with 71643, vertex 9 sees 1,7,x but misses 3,6,0; vertex z sees 0,0 but misses 3,6,7; vertex 0 sees 3,4,x but misses 6,7,9. Now

0 misses 1  (if 0 sees 1 then 0 misses 2 (to avoid a $P_6$ on 063142), and so 05623 is a $C_5$ or $\{1,2,3,4,5,6,0\}$ induces $E_7$);
0 sees 8   (If 0 misses 8 then 0 sees 5 (to avoid a $P_6$ on 045817),
            but then either 02185 or 05623 is a $C_5$),
0 misses 5  (if 0 sees 5 then 05623 is a $C_5$ or 063524 is a $P_6$),
z sees 4   (if z misses 4 then z 0461 is a $C_5$ or z 04617 is a $P_6$),
z misses 1  (if z sees 1 then z 1834 is a $C_5$ or z 60148 is a $P_6$),
9 sees 4   (to avoid a $C_5$ on 9164x),
9 sees 2   (to avoid a $C_5$ on 91234),
9 sees 8   (to avoid a $C_5$ on 04381),
9 sees 5   (to avoid a $P_6$ on 695148),
z misses 2  (to avoid a $P_6$ on 96x142),
z misses 8  (to avoid a $P_6$ on 96x148),
z sees 5   (to avoid a $P_6$ on z 45812),
Figure 4.5.3
0 misses 2  
(to avoid a $C_5$ on $x5620$).

But now $\{x,0,3,2,6,5,8\}$ induces $L_7$. This contradiction justifies Claim 1.2.3, and therefore Claim 1.2.

Claim 1.3: 63142 does not have index three.

Assume the contrary; let $Q = 87892$ be a bypass of 63142 in $\overline{G}$. Thus, by (*), in $G$
vertex 7 sees 1,2,9 but misses 3,6,8; vertex 8 sees 1,2,6 but misses 7,9; vertex 9 sees 1,6,7
but misses 2,4,8. By (*) with 12345, 9 sees 3. Now

8 misses 3  
(to avoid a $P_5$ on $137892$),

8 misses 4  
(to avoid a $C_5$ on $84391$),

8 misses 5  
(to avoid a $C_5$ on $82345$),

9 sees 5  
(to avoid a $P_5$ on $543918$),

7 sees 4  
(if 7 misses 4 then 72345 is a $C_5$ or 827954 is a $P_5$),

7 sees 5  
(to avoid a $P_5$ on $675149$).

But then removing vertex 6 and relabelling vertices 7,8,9 as 6,7,8 respectively gives the
graph in Figure 4.5.2, and we are done by Claim 1.2. This concludes Claim 1.3, which
(finally) concludes Claim 1.

Claim 2: No bad $P_5$ has index two.

Assume the contrary; let 12345 be a bad $P_5$ with bypass $P = 1675$. By (*) with 12345,
6 sees 1,2,7 but misses 3,5; 7 sees 5,4,6 but misses 1,3. Now 7 must see 2; suppose not.
By Claim 1 (with 7 in place of 5), 12347 must extend into a $C_6$, say 123478. But then
(*) is contradicted by 12345 and 8. Thus 7 sees 2; by symmetry, 6 sees 4. The graph
induced by $\{1,\ldots,7\}$ is shown in Figure 4.5.3.

Now note that in $\overline{G}$ 63142 is a bad $P_5$. (Assume the contrary; let 863142 be a $C_6$
in $G$. Then either 84721 is a $C_5$ or 682417 is a $P_5$.)
Claim 2.2: 63142 does not have index two.

Assume the contrary; let $S = 6892$ be a bypass of 63142 in $\overline{G}$. Arguing as in the beginning of Claim 2, in $\overline{G}$ both 8 and 9 see 3 and 4. But then, in $G$, 9 sees 1 but misses 2,3, which contradicts (*) with 12345. This concludes Claim 2.2.

Claim 2.3: 63142 does not have index three.

Assume the contrary; let $S = 68092$ be a bypass of 63142 in $\overline{G}$. Thus, using (*) with 63142, in $G$, 8 sees 1,2,9 but misses 3,6,0; 9 sees 1,6,8 but misses 2,4,0; 0 sees 1,2,6 but misses 8,9. Now

9 sees 3 (if 9 misses 3 then (*) with 12345 is contradicted),
0 misses 3 (to avoid a $\overline{P}_6$ on 138092),
0 misses 4 (to avoid a $C_5$ on 01934),
8 misses 5 (if 8 sees 5 then 12345 is bad $P_5$ with index one),
0 misses 5 (if 0 sees 5 then 12345 is a bad $P_5$ with index one),
9 sees 5 (to avoid a $P_6$ on 019345), and
8 sees 4 (to avoid a $P_6$ on 028954).

Now 83149 extends to a $C_5$ in $\overline{G}$, say 83149z. (Suppose not; in $\overline{G}$, 0 sees 8,9 but misses 1, and so 83149 is a bad $P_5$ with index one, contradicting Claim 1.) Then, in $G$, z misses 2 (to avoid a $\overline{P}_6$ on z83142). But in $\overline{G}$, z sees 2 but misses 1,4, which contradicts (*) with 63142. This concludes Claim 2.3, and also Claim 2.

Claim 3: No bad $P_5$ has index three.

Assume the contrary; let 12345 be a bad $P_5$ with bypass $P = 16785$. Thus 16785 is a chordless path such that 3 misses 6,7,8. Since 12345 has index three, 1 misses 7 and 8, 5 misses 6 and 7; by (*), 6 sees 2 and 8 sees 4.
Figure 4.5.4
If 6 misses 4 then 62345 is a $P_5$ of index at most two (consider 6785) and hence not a bad $P_6$, by Claims 1 and 2; thus there is a $C_5$ of the form 623459, contradicting the assumption that 12345 has index three (consider 1695). Hence 6 sees 4; by symmetry, 8 sees 2. The subgraph of $G$ induced by $\{1,...,8\}$ is shown in Figure 4.5.4 (the vertex 7 may or may not see 2, and may or may see 4).

Now suppose that $\overline{G}$ contains a $C_5$ of the form 631420. Then

0 sees 8 \hspace{1cm} (to avoid a $C_5$ on 01284),

0 sees 7 \hspace{1cm} (to avoid a $C_5$ on 01678),

2 misses 7 \hspace{1cm} (to avoid a $P_6$ on 718602),

and finally 76230 is a $C_5$, a contradiction.

Hence we may assume that 63142 is a bad $P_6$ in $\overline{G}$; by Claims 1 and 2, its index is three. But then we obtain the desired contradiction by forgetting all about 7 and 8 and following the proof of Claim 2.3 (which does not refer at all to vertex 7 of Figure 4.5.3).

This concludes the proof of Claim 3, and the $P_5$ Lemma. $\blacksquare$
The next lemma is a stronger statement than the $L_7$ Lemma, in that it implies that two particular six-vertex induced subgraphs of $L_7$ (and their complements) are forbidden induced subgraphs of graphs in $M$. This lemma will be used in the proof of the $C_6$ Lemma.

**The Stronger Lemma.** If $G$ is a graph in $M$, then $G$ does not contain either

(*) a $P_5$ 12345 and a vertex 6 that sees 1,2,4,5 but misses 3, or

(**) a $P_5$ 12345 and a vertex 6 that sees 2,3 but misses 1,4,5.

**Proof.** To prove (*), note that by the $P_5$ Lemma the $P_5$ 24136 must extend to a $C_6$. Thus there is a vertex 7 that sees 1,3,4 but misses 2,6 in $G$. But this is impossible, since if 7 sees 5 then 23756 is a $C_5$, whereas if 7 misses 5 then $\{1,\ldots,7\}$ induces $L_7$.

To prove (**), note that by the $P_5$ Lemma the $P_5$ 12345 must extend to a $C_6$. Thus there is a vertex 7 that sees 1,5 but misses 2,3,4 in $G$. But this is impossible, since if 7 sees 6 then 34576 is a $C_5$, whereas if 7 misses 6 then 634571 is a $P_6$. \[\]
Figures 4.6.1 and 4.6.2  (top and bottom)
The following lemma describes restrictions on the ways in which vertices of a graph $G$ in $M$ can attach to one of two particular seven-vertex subgraphs of $G$. This result will be used in the $C_6$ Lemma, and also in the Second Extension Lemma.

**The Little Local Lemma.** Let $X = \{r_i, r_j, r_i, s_i, s_j, s_i, v\}$ be a subset of the vertices of a graph $G$ in $M$.

(1A) If $G_X$ is the graph in Figure 4.6.1 and there are vertices $w_i, w_j \in G - X$ such that $w_i$ sees $s_j, s_i$ but misses $r_j, r_i, v$, and $w_j$ sees $s_i, s_j$ but misses $r_i, r_i, v$, then either $w_i$ or $w_j$ sees $s_i, s_j, s_i$ but misses $r_i, r_j, r_i, v$.

(1B) If $G_X$ is the graph in Figure 4.6.1 and there is a vertex $w \in G - X$ such that $w$ sees $s_i, s_i$ but misses $r_j, r_i, v$, then $w$ sees $s_i, s_j, s_i$ but misses $r_i, r_j, r_i, v$.

(2A) If $G_X$ is the graph in Figure 4.6.2 and there is a vertex $w \in G - X$ such that $w$ sees $s_j, s_i$ but misses $r_j, r_i, v$, then $w$ sees $s_i, s_j, s_i$ but misses $r_i, r_j, r_i, v$.

(2B) If $G_X$ is the graph in Figure 4.6.2 and there is a vertex $w$ in $G - X$ such that $w$ sees $s_i, s_i$ but misses $r_j, r_i, v$, then $w$ sees $s_i, s_j, s_i$ but misses $r_i, r_j, r_i, v$.

**Proof.** To prove (1A), assume the contrary. Now $w_i$ must see $r_i$ (if not, then $w_i$ must see $s_i$ to avoid a $P_5$ on $w_i s_i r_i v r_i s_i$, and we are done) and therefore miss $s_i$ (to avoid a $C_5$ on $w_i r_i v r_i s_i$). By symmetry, $w_j$ must see $r_j$ and miss $s_j$. But then $w_i w_j s_i r_i s_j$ is a $C_5$ or $w_i s_j r_i v r_i w_j$ is a $P_6$, a contradiction.

To prove (1B), note that $w$ misses $r_i$ (to avoid a $C_5$ on $w r_i v r_i s_i$) and sees $s_j$ (to avoid a $P_5$ on $w s_i r_j v r_i s_j$).

To prove (2A), note that $w$ misses $r_i$ (to avoid a $C_5$ on $w r_i v r_i s_j$) and sees $s_i$ (to avoid a $P_6$ on $w s_i r_j v r_i s_j$).

To prove (2B), note that $w$ sees $s_j$ (to avoid a $C_5$ on $w s_i s_j r_j s_i$) and thus misses $r_i$ (to avoid a $C_5$ on $w s_j r_i v r_i$).
Figure 4.7
The following lemma describes how certain seven-vertex induced subgraphs (of graphs in $M$) that contain $C_6$ extend to other induced subgraphs. This lemma will be used as the basis case in the proof of Theorem 4.1.

**The $C_6$ Lemma.** Let $X = \{r_i, r_j, r_t, s_i, s_j, s_t, v\}$ be a subset of vertices of a graph $G$ in $M$.

1. If $G_X$ is the graph in Figure 4.7.1A, then there is a vertex $w$ in $G$, such that $G_X \cup \{w\}$ is the graph in Figure 4.7.1B.

2. If $G_X$ is the graph in Figure 4.7.2A, then there is a vertex $w$ in $G$, such that $G_X \cup \{w\}$ is the graph in Figure 4.7.2B.

3. If $G_X$ is the graph in Figure 4.7.3A, then there are vertices $w, x, y$ in $G$, such that $G_X \cup \{w, x, y\}$ is the graph in Figure 4.7.3B.

4. If $G_X$ is the graph in Figure 4.7.4A, then there are vertices $w, x, y$ in $G$, such that $G_X \cup \{w, x, y\}$ is the graph in Figure 4.7.4B, Figure 4.7.4C, or 4.7.4D.

Before proving the lemma, we present a claim which will be used in two of the four cases of the proof. Throughout the claim, (*) and (**) refer back to the Stronger Lemma.

**Claim.** Let 123456 be a $C_6$ in a graph $G$ in $M$, and let 7 be a vertex of $G$ that sees 2, 6 but not 3, 4, 5 (7 may or may not see 1). Then there is a vertex 8 in $G$ that sees 1, 3, 5 but not 2, 6, 7 (8 may or may not see 4).

**Proof of Claim.** Since $G$ is unbreakable, there must be a path from 1 to 3, none of whose vertices sees 7. Let $P$ be any shortest such path. Note that $P$ is chordless.

**Case 1:** $P$ has exactly one interior vertex.

Let $P = 183$. If 8 sees 2 then
8 sees 6 (if 8 misses 6 then 8 misses 5 (to avoid a $C_6$ on 82765), but now 81654 is a $C_6$ or 827654 is a $P_6$),

8 misses 5 (if 8 sees 5 then 8 with 32755 contradicts (*)), and then

if 8 sees 4 then 8 with 34561 contradicts (*), else 83456 is a $C_6$; contradiction.

So 8 misses 2. Now 8 misses 6 (to avoid a $C_6$ on z 3276), and finally 8 sees 5 (to avoid a $P_6$ on 832765). Thus 8 is the desired vertex.

Case 2: P has exactly two interior vertices.

Let $P = 1xy3$. If x sees 5, then we are in Case 1: switch 2 with 6 and 2 with 5. Hence we may assume that x misses 5. Then x must see 6 (if not, x 1654 is a $C_6$ or x 16543 is a $P_6$). Thus x must see 2 (else x with 56123 contradicts (**)).

If x sees 4 then, by (1) with x in place of 7, some vertex 8 sees 1,3,4,5 and misses 2,6,7; note that 8 misses 7 (to avoid a $C_6$ on 87264).

Hence we may assume that x misses 4. Applying the argument of Case 1 with x in place of 1 and with y in place of 8, we conclude that y sees x,3,5 and misses 2,6,7. Now by (1) with y in place of 4 and with x in place of 7, some vertex 8 sees 1,3,4,5 and does not see 2,6,7; note that 8 misses 7 (to avoid a $C_6$ on 8762xy).

Case 3: P has exactly three interior vertices.

Let $P = 1xyz3$. As in Case 2, we may assume that x misses 5, sees 6, sees 2 and misses 4. By Case 2, there is a vertex w that sees x,3,5 and not 2,6,7. If w sees 1, then we may set $8 = w$ with x in place of 1; hence we may assume that w misses 1. By (1) with w in place of 4 and with x in place of 7, some vertex 8 sees 1,3,6,5 and misses 2,6,7; note that 8 misses 7 (to avoid a $C_6$ on 8762xw). This concludes the proof of the Claim. $\blacksquare$
Proof of the C₉ Lemma. To prove (1), by the P₅ Lemma the P₅ rᵢ rᵢ sᵢ sᵢ v must extend to a C₉; suppose that some vertex w sees sᵢ, sᵢ, rᵢ but misses rᵢ, v. Then w does not see rᵢ (else w rᵢ v rᵢ sᵢ is a C₁) and w sees sᵢ (else w sᵢ rᵢ sᵢ rᵢ is a C₉). Thus (1) is proved.

To prove (2), by the P₅ Lemma the P₅ sᵢ rᵢ v rᵢ sᵢ must extend to a C₉; thus there is a vertex wᵢ that sees sᵢ, sᵢ but misses rᵢ, rᵢ, v. Similarly, the P₅ sᵢ rᵢ v rᵢ sᵢ must extend to a C₉; thus there is a vertex wᵢ that sees sᵢ, sᵢ but misses rᵢ, rᵢ, v. Now, by (1A) of the Little Local Lemma, it follows that either wᵢ or wᵢ is the desired vertex w.

To prove (3), by the Claim (with vertex v and the C₉ sᵢ rᵢ sᵢ rᵢ sᵢ rᵢ in place of 7 and the C₉ 123456 respectively) there is a vertex v that sees sᵢ, sᵢ, sᵢ but misses rᵢ, rᵢ, v. Similarly, (by the Claim with vertex sᵢ and the C₉ v rᵢ sᵢ rᵢ sᵢ rᵢ ) there is a vertex v that sees v, sᵢ, sᵢ but misses rᵢ, rᵢ, sᵢ. Next, (by the Claim with vertex v and the C₉ rᵢ sᵢ rᵢ v rᵢ sᵢ ) there is a vertex v that sees rᵢ, rᵢ, rᵢ and misses v, sᵢ, sᵢ.

Now

x misses y (to avoid a C₅ on x sᵢ rᵢ v y),
x sees rᵢ (to avoid a P₅ on rᵢ sᵢ x sᵢ rᵢ v),
y sees rᵢ (to avoid a P₅ on rᵢ sᵢ y v rᵢ sᵢ),
w sees sᵢ (to avoid a C₅ on w rᵢ x sᵢ rᵢ),
w misses y (to avoid a C₅ on sᵢ w y sᵢ x),
w sees v (to avoid a C₅ on w rᵢ v rᵢ), and (3) is proved.

To prove (4), argue as in the beginning of the proof of (3): there are vertices x, y such that vertex x sees sᵢ, sᵢ, sᵢ but misses rᵢ, rᵢ, v, and vertex y sees v, sᵢ, sᵢ but misses rᵢ, rᵢ, sᵢ. Note that x sees y (to avoid a C₅ on sᵢ v y sᵢ x). There are three cases to consider.
Case 1: \( r_t \) misses \( x \).

Applying the Claim to the \( G \) on \( r_t \, r_j \, v \, r_i \, s_j \) and to vertex \( x \), we find a vertex \( w \) that sees \( r_t, r_i, r_j \) but misses \( x, s_j, s_i \). Now

\[
\begin{align*}
& w \text{ misses } s_i & (\text{to avoid a } C_5 \text{ on } s_t \, w \, r_t \, s_i \, x), \\
& w \text{ misses } y & (\text{to avoid a } C_5 \text{ on } w \, y \, x \, s_i \, r_i), \\
& y \text{ sees } r_i & (\text{to avoid a } P_6 \text{ on } y \, s_i \, r_i \, w \, r_i \, s_t), \\
& w \text{ sees } v & (\text{to avoid a } P_6 \text{ on } v \, r_j \, w \, r_i \, s_j \, x).
\end{align*}
\]

and the graph induced by \( \{s_i, r_t, s_j, r_i, s_t, r_j, v, x, y, w\} \) is that shown in Figure 4.7.4B.

Case 2: \( r_t \) misses see \( y \).

Applying the Claim to the \( G \) on \( r_t \, s_i \, r_j \, s_i \, r_i \, s_j \) and to vertex \( y \), we find a vertex \( w \) that sees \( r_t, r_i, r_j \) but misses \( s_j, s_i, y \). Now

\[
\begin{align*}
& w \text{ misses } v & (\text{to avoid a } C_5 \text{ on } v \, w \, r_t \, s_i \, y), \\
& w \text{ misses } x & (\text{to avoid a } C_5 \text{ on } w \, x \, v \, r_i), \\
& x \text{ sees } r_i & (\text{to avoid a } P_6 \text{ on } x \, s_i \, r_i \, w \, r_i \, v), \\
& w \text{ sees } s_t & (\text{to avoid a } P_6 \text{ on } s_t \, r_j \, w \, r_t \, s_j \, y),
\end{align*}
\]

and the graph induced by \( \{s_i, r_t, s_j, r_i, s_t, r_j, v, x, y, w\} \) is that shown in Figure 4.7.4C.

Case 3: \( r_t \) sees \( x \) and \( y \).

Applying (2) to the \( G \) on \( s_t \, v \, x \, r_i \, y \, s_i \) and to vertex \( r_t \), we find a vertex \( w \) that sees (in \( G \)) \( s_i, r_i, r_t, v \) and does not see \( s_j, x, y \). Now

\[
\begin{align*}
& w \text{ misses } s_i & (\text{to avoid a } C_5 \text{ on } w \, s_i \, x \, s_j \, r_i), \\
& w \text{ sees } r_j & (\text{to avoid a } P_6 \text{ on } w \, r_t \, s_j \, x \, s_i \, r_j),
\end{align*}
\]

and the graph induced by \( \{s_t, r_t, s_j, r_i, s_i, r_j, v, x, y, w\} \) is that shown in Figure 4.7.4D.

This concludes the proof of the \( C_6 \) Lemma. ■
4.3 Strong Mirror Graphs

It is easier to prove Theorem 4.1 by dealing only with a certain subclass of mirror graphs that includes all unbreakable mirror graphs, rather than by dealing with all mirror graphs. This subclass is the class of "strong mirror graphs"; we present a formal definition shortly. It turns out that a mirror graph is unbreakable if and only if it is a strong mirror graph. As we did with $L_8$ and $L_9$, we will postpone the proof of unbreakability, i.e. the "if" part of the previous statement, until Section 4.4.

We shall say that a $P_4$-free graph $G$ is strong unless (and only unless) $G$ or $\overline{G}$ has precisely two components and one of these components is a singleton. The following lemma is a useful tool for working with strong $P_4$-free graphs. The graph $2K_2$ referred to in the lemma is the graph with two components, each of which is a single edge.

**The Rip-Off Lemma.** Let $G$ be a strong $P_4$-free graph with at least four vertices such that neither $G$ nor $\overline{G}$ is $2K_2$. Then $G$ contains twins $x,y$ such that $G - x$ and $G - y$ are strong $P_4$-free graphs. Furthermore, if $G$ has an isolated vertex $z$, then we can choose $x,y$ both distinct from $z$.

**Proof.** First, let us prove only that $G$ contains twins $c,d$ such that both $G - c$ and $G - d$ are strong $P_4$-free graphs. Let $a,b$ be twins in $G$. Since $G - a$ and $G - b$ are isomorphic, we may assume that $G - a$ is not strong (otherwise we are done by setting $c = a$, $d = b$). Replacing $G$ by $\overline{G}$ if necessary, we may assume that $G - a$ has precisely two components and that one of them is a singleton. Note that the singleton is $b$ (else $G$ would not be strong); call the other component $Q$; observe that $Q$ is a component of $G$. Now let $c,d$ be any twins in $Q$.

To complete the proof, assume that one of $c,d$ is isolated in $G$ (otherwise we can set $x = c$ and $y = d$). Then both $c$ and $d$ are isolated in $G$. If $G$ has no edges at
Figure 4.8. A $P_4$-free graph and its decomposition tree
all then any two vertices \( x, y \) distinct from \( c \) and \( d \) will do; else \( G \) has a big component \( Q \) and any twins \( x, y \) in \( Q \) will do. \( \Box \)

Strong mirror graphs are defined as follows: start with the definition of a mirror graph, insist that the \( P_4 \)-free graph \( G_R \) be strong, and specify exactly which couples of the partition induce edges of the graph (that is, for which couples \( \{ r_j, s_j \} \) the vertices \( r_j \) and \( s_j \) are adjacent). This specification is in the form of a certain \( 0 \)-\( 1 \) function \( f \); this function is defined in terms of a decomposition of \( P_4 \)-free graphs that follows from repeatedly applying Seinsche's theorem. (Recall Seinsche's theorem from Chapter 1: if a \( P_4 \)-free graph has at least two vertices, then either the graph or its complement is disconnected.)

We now present a recursive definition of a graph \( DT(G) \) whose vertices correspond to subsets of vertices of another graph \( G \). In order to avoid ambiguity, we will refer to the vertices of \( DT(G) \) as nodes. The decomposition tree \( DT(G) \) of a \( P_4 \)-free graph \( G \) is the rooted tree such that:

1. if \( G \) has only one vertex \( v \), then the root of \( DT(G) \) is the vertex \( v \), and there are no other nodes in \( DT(G) \), and

2. if \( G \) has more than one vertex, then the root of \( DT(G) \) is the set of all vertices of \( G \), and the nodes adjacent to the root are \( DT(G_1), \ldots, DT(G_k) \), where \( G_1, \ldots, G_k \) are the induced subgraphs of \( G \) that correspond to the components of whichever of \( G \) or \( \overline{G} \) is disconnected.

A \( P_4 \)-free graph and its decomposition tree are shown in Figure 4.8. Note that every vertex of \( G \) is a leaf of \( DT(G) \). Also, every leaf of \( DT(G) \) is a vertex of \( G \), and every node of \( DT(G) \) that is not a leaf is a subset of at least two of the vertices of \( G \). Note also that \( DT(G) \) is identical to \( DT(\overline{G}) \).
Figure 4.9. A strong mirror graph
We need one more definition before we can define the 0-1 function \( f \). Let \( G \) be a \( P_4 \)-free graph with at least two vertices. For every vertex \( v \) of a \( P_4 \)-free graph \( G \) with at least two vertices, define the parent \( P(G,v) \) to be the parent of \( v \) in \( DT(G) \), (i.e. the node of \( DT(G) \) adjacent to the leaf \( v \)). For example, with respect to the \( P_4 \)-free graph \( G \) shown in Figure 4.8, the parent of 1 is the root of \( DT(G) \) (namely, the set of all vertices of \( G \)), the parent of 2, 3, and 4 is the node \( \{2,3,4\} \), the parent of 5 and 8 is the node \( \{5,6,7,8\} \), and the parent of 6 and 7 is the node \( \{6,7\} \).

Now define the function \( f(G,v) \) so that
\[
f(G,v) = 0 \quad \text{if } G_{P(G,v)} \text{ is disconnected, and}
\]
\[
f(G,v) = 1 \quad \text{if } G_{P(G,v)} \text{ is connected.}
\]
Note that \( v \) is a singleton in whichever of \( G_{P(G,v)} \) or \( \overline{G}_{P(G,v)} \) is disconnected. For the graph \( G \) shown in Figure 4.8, \( f(G,v) = 0,1,1,1,0,0,1,1 \) for \( v = 1,2,\ldots,8 \) respectively.

Now that \( f(G,v) \) is defined, we can formally define strong mirror graphs. A partition \([R,S]\) of the vertices of a graph \( G \) is called a strong mirror partition if conditions (1) and (2) of the definition of a mirror partition are satisfied, and if
\[
(3) \quad G_R \text{ is a strong } P_4\text{-free graph, and}
\]
\[
(4) \quad r_i \text{ sees } s_j \quad \text{if and only if} \quad f(G_R,r_i) = 1, \quad \text{for all } r_i \in R.
\]
A graph with a strong mirror partition is a strong mirror graph. A strong mirror graph is shown in Figure 4.9.
The classes of $P_4$-free graphs and murky graphs are self-complementary. We now show that the same is true for the classes of mirror graphs and strong mirror graphs.

**The Complement Lemma.** Let $[R,S]$ be a (strong) mirror partition of $G$. Then the partition $[R,S]$, with vertices labelled as in the partition of $G$, is a (strong) mirror partition of $\overline{G}$.

**Proof.** The conditions (1), (2), (3), (4) mentioned in the proof refer to the definitions of mirror partition and strong mirror partition.

Let $G$ be a graph with mirror partition $[R,S]$. Since the complement of a $P_4$-free graph is $P_4$-free, the partition $[R,S]$ of $\overline{G}$ satisfies condition (1).

Let $r_m$ and $r_p$ be any two vertices of $R$; $r_m$ sees $r_p$ in $\overline{G}$ if and only $r_m$ misses $r_p$ in $G$. From (2) it follows that in $G$

\[
\begin{align*}
    r_m \text{ misses } r_p & \quad \text{if and only if} \quad s_m \text{ misses } s_p \\
    r_m \text{ sees } s_p & \quad \text{if and only if} \quad s_m \text{ sees } r_p
\end{align*}
\]

Thus in $\overline{G}$

\[
\begin{align*}
    r_m \text{ sees } r_p & \quad \text{if and only if} \quad s_m \text{ sees } s_p \\
    r_m \text{ misses } s_p & \quad \text{if and only if} \quad s_m \text{ misses } r_p
\end{align*}
\]

and (2) holds for the partition $[R,S]$ of $\overline{G}$. Thus $[R,S]$ is a mirror partition of $\overline{G}$.

Now assume that $[R,S]$ is a strong mirror partition of $G$; we will prove that it is also a strong mirror partition of $\overline{G}$. By the previous argument we need only prove that (3) and (4) hold for $[R,S]$ with respect to $\overline{G}$. But (3) holds trivially. To see that (4) holds, note that $DT(H) = DT(\overline{H})$ for any $P_4$-free graph $H$; thus $H_{P(H,v)}$ is the complement of $\overline{H}_{P(H,v)}$, and so (if $H$ has at least two vertices) $f(H,v) + f(\overline{H},v) = 1$.

Now set $H = G_R$, and use the fact that (4) holds for $[R,S]$ with respect to $G$. $\blacksquare$
The graph shown in Figure 4.9 is a strong mirror graph, since the partition suggested by the drawing is a strong mirror partition. (Partition the vertices into the "upper set" and the "lower set"; the couples are the pairs of vertically aligned vertices. Note that the subgraphs induced by "upper set" and "lower set" respectively are isomorphic to the graph shown in Figure 4.8.) On the other hand, the partition suggested by the drawing of the mirror graph in Figure 4.2 is not a strong mirror partition (in fact this graph has no strong mirror partition). In Section 4.4 we will say more about which mirror graphs have strong mirror partitions. First, however, we wish to prove Theorem 4.1. With this goal in mind, we state two results concerning the function $f$.

The Localization Lemma. Let $G$ be a $P_4$-free graph and let $H$ be a homogeneous set in $G$. Then $f(G,x) = f(G_H,x)$, for all $x \in H$.

Proof. Consider an arbitrary vertex $x$ in $H$. The Complement Lemma allows us to assume that $f(G,x) = 0$. We may assume that $f(G_H,x) = 1$, for otherwise we are done. Let $A$ be the parent of $x$ in $DT(G)$; since $f(G,x) = 0$, vertex $x$ is isolated in $A$. Let $B$ be the parent of $x$ in $DT(G_H)$; since $f(G_H,x) = 1$, vertex $x$ sees all the remaining vertices in $B$. It follows that the intersection of $A$ and $B$ contains only $x$. Since both $A$ and $B$ have at least two vertices, there is some vertex $a \in A - B$, and some vertex $b \in B - A$.

Note that $A$ is homogeneous in $G$ and that $B$ (being homogeneous in $G_H$) is also homogeneous in $G$. Since $a$ misses $x$, it must miss all of $B$; in particular, $a$ misses $b$. Since $b$ sees $x$, it must see all of $B$; in particular, $b$ sees $a$; contradiction.
A special case of the Localization Lemma asserts that \( f(G,x) = f(G,y) = 1 \) whenever \( x,y \) are adjacent twins and that \( f(G,x) = f(G,y) = 0 \) whenever \( x,y \) are non-adjacent twins.

The following lemma is also concerned with \( f \) and with twins.

**The Twin Lemma.** Let \( G \) be a \( P_4 \)-free graph with at least three vertices. If \( x,y \) are twins in \( G \) then \( f(G,x) = f(G-x, z) = f(G-y, z) \), for all \( z \) in \( G - \{x,y\} \).

**Proof.** Argue by induction on \( |G| \). Since \( f(G,x) + f(\overline{G}, x) = 1 \) for all \( P_4 \)-free graphs \( G \), we may assume that \( G \) is disconnected: its vertices can be partitioned into non-empty disjoint sets \( S_1, S_2 \), so that no edge has one vertex in each \( S_i \). If \( x \) and \( y \) belong to distinct \( S_i \)'s, then each vertex distinct from both \( x \) and \( y \) misses at least one of them, and therefore it misses both; in that case, we can redefine \( S_1, S_2 \) by setting \( S_1 = \{x,y\} \) and letting \( S_2 \) consist of all the remaining vertices.

Hence we may assume that \( x,y \in S_1 \). To prove the lemma for all \( z \) in \( S_1 \), distinct from both \( x \) and \( y \), we may assume that \( |S_1| \geq 3 \) (else there is nothing to prove); the induction hypothesis guarantees that

\[ f(G_{S_1}, z) = f(G_{S_1-x}, z) = f(G_{S_1-y}, z) \] whenever \( z \in S_1 \), \( z \neq x,y \);

the Localization Lemma guarantees that

\[ f(G, z) = f(G_{S_1}, z), \quad f(G-x, z) = f(G_{S_1-x}, z), \quad f(G-y, z) = f(G_{S_1-y}, z). \]

Now combining these two sets of equalities yields the desired conclusion.

To prove the lemma for all \( z \in S_2 \), we may assume that \( |S_2| \geq 2 \) (else \( f(G,x) = f(G-x,z) = f(G-y,z) = 0 \) for the singleton \( z \) in \( S_2 \), and we are done). Clearly, \( S_3 \) is a homogeneous set of \( G \), \( G-x \), and \( G-y \); now the Localization Lemma implies the desired conclusion. \( \blacksquare \)
Having built up a repertoire of results concerning \( f \), we are able to present some lemmas concerning strong mirror graphs.

**The Reduction Lemma.** Let \( G \) be a strong mirror graph with at least eight vertices such that neither \( G_R \) nor \( \overline{G}_R \) is \( 2K_2 \). Then there are twins \( r_i, r_j \) in \( G_R \) such that either

(a) \( [R-r_i,S-s_i] \) is a strong mirror partition of \( G - \{r_i, s_i\} \),
(b) \( [R-r_i,S-s_j] \) is a strong mirror partition of \( G - \{r_i, s_j\} \), and

\[
f(G_R - r_i, r_j) = f(G_R - r_j, r_i) = f(G_R, r_i) = f(G_R, r_j),
\]

or

\[
f(G_R - r_i, r_j) = f(G_R - r_j, r_i) \neq f(G_R, r_i) = f(G_R, r_j).
\]

In all cases, all sets \( \{r_k, s_k\} \) with \( k \neq i, j \) are couples of these strong mirror partitions.

Furthermore, if \( G_R \) has an isolated vertex \( r_i \), then we can choose \( i, j \) both distinct from \( i \).

**Proof.** By the Rip-Off Lemma, we find twins \( r_i, r_j \) in \( G_R \) such that \( G_R - r_i \) and \( G_R - r_j \) are strong \( P_4 \)-free graphs, and such that, for any given isolated vertex of \( r_i \), both \( i, j \) are distinct from \( i \). By the Twin Lemma,

\[
f(G_R - r_i, r_k) = f(G_R - r_j, r_k) = f(G_R, r_k) \text{ whenever } k \neq i, j.
\]

Note that \( G_R - r_i, G_R - r_j, G_S - s_i, G_S - s_j \) are all isomorphic and that

\[
f(G_R - r_i, r_j) = f(G_R - r_j, r_i).
\]

In addition, note that

\[
f(G_R, r_i) = f(G_R, r_j) = 1 \text{ if } r_i \text{ sees } r_j,
\]

\[
f(G_R, r_i) = f(G_R, r_j) = 0 \text{ if } r_i \text{ misses } r_j
\]

(use the Localization Lemma with \( G = G_R, H = \{r_i, r_j\} \)). Hence (a) holds if

\[
f(G_R - r_i, r_j) = f(G_R, r_j),
\]

and (b) holds in the other case. \( \blacksquare \)
One difficulty that must be overcome in proving theorems that concern strong mirror graphs is that a strong mirror graph can have more than one strong mirror partition. For example, the strong mirror graphs shown in Figures 4.10, 4.11, and 4.12 are isomorphic and yet have different strong mirror partitions. The following two lemmas show how this non-uniqueness can be exploited. In particular, the first of these lemmas shows that under certain hypotheses it is possible to "repartition" a strong mirror graph, i.e. find some other strong mirror partition of the graph. The second lemma shows that any given strong mirror graph has a strong mirror partition that "isolates" any given vertex of the graph.

The Repartitioning Lemma. Let $G$ be a strong mirror graph with a strong mirror partition $[R, S]$, and suppose that whichever of $G_R$ or $\overline{G}_R$ is disconnected has some big component. Let $R_1$ be the set of vertices of such a component and let $S_1$ be the set of mates of vertices in $R_1$. Define $R' = R_1 + S - S_1$, and $S' = S_1 + R - R_1$. Then the partition $[R', S']$ in which the couples are the same as the couples of $[R, S]$ is a strong mirror partition of $G$.

Proof of Lemma. Label the vertices of $R'$ and $S'$ so that couples of $[R', S']$ are couples of $[R, S]$, i.e. let $r_i' = r_i$ for all $r_i$ in $R_1$ and let $r_j' = s_j$ for all $s_j$ in $S - S_1$; let $s_i' = s_i$ for all $s_i$ in $S_1$ and let $s_j' = r_j$ for all $s_j$ in $R - R_1$. To prove the lemma it suffices to confirm that the following four properties hold.

1. $G_R'$ and $G_S'$ are $P_4$-free,
2. $r_i'$ sees $r_j'$ if and only if $s_i'$ sees $s_j'$ if and only if $r_i'$ misses $s_j'$ if and only if $s_i'$ misses $r_j'$, for all $i \neq j$.
3. $G_R'$ and $G_S'$ are strong $P_4$-free graphs,
4. $r_j'$ sees $s_j'$ if and only if $f(G_R', r_j') = 1$, for all $r_j'$ in $R'$. 

Both $G_{R_1}$ and $G_{S - S_1}$ are $P_4$-free, and there are either no edges or all edges between $G_{R_1}$ and $G_{S - S_1}$; thus $G_{R_1'}$ is $P_4$-free. By symmetry, so is $G_{S_1'}$, and (1) holds.

To see that (3) holds, consider first the case in which $G_{R_1'}$ is disconnected; note that $G_{R_1'}$ is disconnected. Since $R_1$ induces a big component in $G_{R_1}$, $G_{R_1'}$ has at least three components (at least two are induced by $R_1$, and at least one is induced by $S - S_1$), and so $G_{R_1'}$ is a strong $P_4$-free graph. Similarly, in the case where $G_{S_1'}$ is disconnected, $G_{S_1'}$ is disconnected and has at least three components, and (3) holds.

Note that (2) is equivalent to the following:

for every two distinct couples $X = \{a, b\}$ and $Y = \{c, d\}$,

each vertex of $X$ sees exactly one vertex of $Y$.

Since this property holds with respect to the partition $[R, S]$, and since the couples of $[R, S]$ are the same as the couples of $[R', S']$, it follows that this property holds with respect to the partition $[R', S']$. Thus (2) holds.

Finally, to show that (4) holds, let $r_j'$ be any vertex in $R_1$. Note that $R_1$ is a homogeneous set of both $G_{R_1'}$ and $G_{R_1'}$; by the Localization Lemma, $f(G_{R_1'}, r_j') = f(G_{R_1}, r_j')$ and $f(G_{R_1}, r_j) = f(G_{R_1}, r_j)$. Since $r_j' = r_j$ and $G_{R_1'} = G_{R_1}$, it follows that $f(G_{R_1'}, r_j') = f(G_{R_1}, r_j)$. This, together with the fact that (by the repartitioning) $r_j'$ sees $s_j'$ if and only if $r_j$ sees $s_j$, and the fact that (since $[R, S]$ is a strong mirror partition) $r_j$ sees $s_j$ if and only if $f(G_{R_1}, r_j) = 1$, imply that $r_j'$ sees $s_j'$ if and only if $f(G_{R_1'}, r_j') = 1$. Thus (4) holds in this case.

In the other case, $r_j' \in S - S_1$. Since $G_{R_1}$ is a strong $P_4$-free graph, $|S - S_1| \geq 2$, and $S - S_1$ is a homogeneous set of both $G_{R_1'}$ and $G_{S_1}$. Now a similar argument to that of the previous case implies that (4) holds in this case as well. \qed
Figure 4.10 shows a strong mirror graph; the partition suggested by the drawing (i.e. "upper part" and "lower part") is a strong mirror partition. The graph in Figure 4.11 can be obtained by repartitioning the graph in Figure 4.10 as follows: let $R_1$ be the leftmost component of the "upper part" of the graph in Figure 4.10, and repartition as described in the Repartition Lemma. Similarly, the graph in Figure 4.12 can be obtained by repartitioning the graph in Figure 4.11.

The Isolation Lemma. Let $G$ be a graph with a strong mirror partition $[R,S]$, and let $v$ be any vertex of $G$. Then there is a strong mirror partition $[R',S']$ of $G$ such that the couples of $[R',S']$ are the couples of $[R,S]$ and such that $v$ is a singleton in whichever of $G_R$, or $\overline{G_R}$, is disconnected.

Proof of Lemma. Assume that $G_R$ is disconnected (the following argument holds if $G_R$ is replaced with $\overline{G_R}$). Let $R_1$ be the set of vertices of $R$ in the component of $G_R$ that contains $v$. The proof is by induction on $|R_1|$.

If $|R_1| = 1$, then $[R,S]$ is the desired partition. Suppose then that $|R_1| \geq 2$. Let $R'$ and $S'$ be as defined in the Repartitioning Lemma. Consider the strong mirror partition $[R',S']$ of $G$. Note that $\overline{G_{R'}}$ is disconnected and has at least three components. Let $R_2'$ be the set of vertices of $R'$ induced by the component of $\overline{G_{R'}}$ that contains $v$. Since $R_2'$ is a proper subset of $R_1$, $|R_2'| < |R_1|$. The lemma now follows by inductive hypothesis and the Repartitioning Lemma. [1]

The Isolation Lemma is illustrated by the graphs shown in Figures 4.10, 4.11 and 4.12. Let $v$ be the upper leftmost vertex in the graph in Figure 4.10. Figure 4.11 and Figure 4.12 show the sequence of two repartitions that isolate $v$. (The vertex $v$ appears as the upper leftmost vertex in all three drawings.)
Figure 4.11
The following two lemmas describe restrictions on how vertices in graphs in $M$ can "attach" to strong mirror subgraphs.

**The Zero-Two Lemma.** *Let $H$ be a graph in $M$, let $G$ be a strong mirror subgraph of $H$, and let $v$ be a vertex of $H - G$ that is partial on $G$. If $v$ is universal or null on some couple $\{r_i, s_i\}$ of a strong mirror partition of $G$, then $v$ is a twin of one of $r_i, s_i$ with respect to $G - \{r_i, s_i\}$.*

**Proof.** Argue by induction on the number of vertices in $G$. By the Complement Lemma and the Isolation Lemma, we may assume that $r_i$ is isolated in $G_R$.

If $G$ has precisely six vertices then $r_i s_i r_j s_i r_i r_j$ is a $C_6$. There are two cases.

Case 1: $v$ is null on $\{r_i, s_i\}$.

Since $v$ sees at least one vertex of $G$, by swapping $R$ and $S$ if necessary, and also $i$ and $j$, we may assume that $v$ sees $s_i$. Now $v$ must see $s_j$ (if not, either $v s_i r_i s_j r_i$ is a $C_5$ or $v s_i r_i s_j r_i s_i$ is a $P_3$). This implies that $v$ misses $r_j$ (if $v$ sees $r_j$, then either $v s_j r_i s_i r_j$ is a $C_5$ or $\{v, r_i, s_i, r_i, s_i, r_j, s_j\}$ induces $L_7$); by symmetry, $v$ also misses $r_i$.

Now observe that $v$ is a twin with respect to $r_i$ of $G - \{r_i, s_i\}$.

Case 2: $v$ is universal on $\{r_i, s_i\}$.

Since $v$ misses at least one vertex of $G$, by swapping $R$ and $S$ if necessary, and also $i$ and $j$, we may assume that $v$ misses $s_i$. Now $v$ sees $r_j$ (to avoid a $C_5$ on $v r_i s_i r_j s_i$), misses $s_j$ (else $v$ and the $P_3 s_j r_i s_i r_j s_i$ contradict (**) of the Stronger Lemma), and finally sees $r_i$ (to avoid a $C_5$ on $v r_i s_j r_i s_i$). Now $v$ is a twin with respect to $s_i$ of $G - \{r_i, s_i\}$.

If $G$ has at least eight vertices then (since $r_i$ is isolated in $G_R$) neither $G_R$ nor $G_R$ is $2K_2$, and so we can apply the Reduction Lemma. Let $r_i, r_j$ be as in the
conclusion of the Lemma; set \( G_i = G - \{r_i, s_i\} \), \( G_j = G - \{r_j, s_j\} \) in case (a) and \( G_i = G - \{r_i, s_j\} \), \( G_j = G - \{r_j, s_i\} \) in case (b). By the induction hypothesis, there is a vertex \( w_i \in \{r_i, s_i\} \) such that \( v \) is a twin of \( w_i \) with respect to \( G_i - \{r_i, s_i\} \) and there is a vertex \( w_j \in \{r_j, s_i\} \) such that such that \( v \) is a twin of \( w_j \) with respect to \( G_j - \{r_i, s_i\} \). We need only prove that \( w_i = w_j \).

Assume the contrary: \( w_i \neq w_j \). Now \( w_i \) and \( w_j \) are anti-twins in \( G \). However, \( v \) is a twin of both with respect to the non-empty graph \( G - \{r_i, s_i, r_j, s_j, r_i, s_i\} \), a contradiction. This concludes the proof of the Zero-Two Lemma.

The Attachment Lemma. Let \( H \) be a graph in \( M \), let \( G \) be a strong mirror subgraph of \( H \), and let \( v \) be a vertex of \( H - G \) that is partial on \( G \). Then either

(i) there is a strong mirror partition \([R, S]\) of \( G \) such that \( v \) is universal on \( R \) and null on \( S \), or

(ii) in every strong mirror partition \([R, S]\) of \( G \) there is a couple \( \{r_i, s_i\} \) such that \( v \) is a twin of one of \( r_i, s_i \) with respect to \( G - \{r_i, s_i\} \).

Proof. We may assume that (ii) does not hold; now the Zero-Two Lemma guarantees the existence of a strong mirror partition \([R, S]\) of \( G \) such that \( v \) is partial on every couple \( \{r_i, s_i\} \).

First we claim that

(1) if \( G_R \) has at least three components then \( v \) is partial on at most two of them.

To justify this claim, assume the contrary: \( G_R \) has components \( R_1, R_2, R_3 \) (and possibly others) such that \( v \) is partial on \( R_1 \) and \( R_2 \). Let \( S_1, S_2, S_3 \) be the corresponding components of \( G_S \). Now there are adjacent vertices \( a \) and \( z \) in \( R_1 \), such that \( v \) sees \( z \) and misses \( a \); let \( b \) denote the mate of \( z \). Now \( a \in R_1, b \in S_1 \), and
$a, b, v$ are pairwise non-adjacent. By symmetry, there are vertices $c$ and $d$ such that $c \in R_2, d \in S_2$ and such that $c, d, v$ are pairwise non-adjacent.

Finally, let $\{x, y\}$ be a couple with $x \in R_3, y \in S_3$. Swapping $R$ and $S$ if necessary, we may assume that $v$ sees $x$. Now we wish to find a vertex $z$ in $S_3$ that misses $x$. If $R_3 = \{x\}$ then $f(G_R, x) = 0$, and so we may set $z = y$; else let $z$ be the mate of any neighbour of $x$ in $R_3$. Now observe that $azcbzd$ is a $C_6$. Since $v$ sees $x$ and misses $a, b, c, d$, either $vzdaz$ is a $C_6$ or $vzdacz$ is a $P_6$, a contradiction.

Next we claim that

(2) $G_R$ has no components $R_1, R_2, R_3$ such that $v$ is

partial on $R_1$, universal on $R_2$, and null on $R_3$.

To justify this claim, assume the contrary. As in the proof of (1), we find a vertex $a$ in $R_1$ and $b$ in $S_1$ such that $a, b, v$ are pairwise non-adjacent. Now let $c$ be any vertex in $R_2$. There is a vertex $d$ in $S_2$ that misses $c$: if $R_2 = \{c\}$ then $f(G_R, c) = 0$, and we may let $d$ be the mate of $c$, else we may let $d$ be the mate of any neighbour of $c$ in $R_2$. Finally, let $e$ be any vertex in $R_3$. Note that $v$ is null on $S_3$; it follows that $vcbeda$ is a $P_6$, a contradiction.

Finally, replacing $H$ by $\overline{H}$ if necessary, we may assume that $G_R$ is disconnected.

Let us distinguish between two cases.

Case 1: $v$ is partial on no component of $G_R$.

In this case, let $R_1$ be the set of neighbours of $v$ in $R$, and let $S_1$ be the set of non-neighbours of $v$ in $S$. Note that $|R_1| \geq 2$ (else $R_1 = \{r_i\}$ and $v$ is a twin of $r_i$ with respect to $G - \{r_i, e_i\}$). Note that $R_1$ and $R - R_1$ are homogeneous in $G_R$; by the Localization Lemma, $|R_1 + S - S_1, S_1 + R - R_1|$ is a strong mirror partition of $G$.

Since $v$ is universal on $R_1 + S - S_1$ and null on $S_1 + R - R_1$, property (i) holds.
Case 2: \( v \) is partial on some component of \( G_R \).

By (1), \( v \) is partial on precisely one component \( R_1 \) of \( G_R \). We shall argue by induction on \( |R_1| \). By (2), \( v \) is universal or null on \( R - R_1 \). Note that \( |R - R_1| \geq 2 \) (because \( G_R \) is strong); hence \( R_1 \) and \( R - R_1 \) are homogeneous in \( G_R \). Set \( R' = R_1 + S - S_1 \), \( S' = S_1 + R - R_1 \). By the Localization Lemma, \( [R', S'] \) is a strong mirror partition of \( G \). Note that \( R' \) induces a disconnected subgraph of \( \overline{G} \), and so does \( R_1 \). By (2), \( v \) is partial on at most one component of \( \overline{G}_{R_1} \). If \( v \) is partial on precisely one such component then we are done by the induction hypothesis applied to the mirror partition \( [R', S'] \) of \( \overline{G} \); if \( v \) is partial on no such component then we are done because the mirror partition \( [R', S'] \) of \( G \) satisfies the hypothesis of Case 1. This concludes the proof of the Attachment Lemma. \( \square \)
Figure 4.13
The following two lemmas are both statements of the following form: suppose that $G$ is a strong mirror subgraph of a graph $H$ in $M$, and suppose that $v$ is some vertex that attaches to $G$ in a certain way; then there is another vertex (or there are other vertices) in $G$ that attach to $H + v$ in another certain way. These lemmas are the last two before the proof of Theorem 4.1.

The First Extension Lemma. Let $G$ be a strong mirror subgraph of a graph $H$ in $M$, let $[R,S]$ be a strong mirror partition of $G$, let $G_R$ be disconnected, and let $v$ be a vertex in $H - G$ that is universal on $R$ and null on $S$. Then there is a vertex $w$ in $H - G$ that misses $v$, is universal on $S$, and null on $R$.

Proof. We shall argue by induction on the number of vertices in $G$. If $G$ has precisely six vertices then it is a $C_6$ and the desired conclusion follows by (2) of the $C_6$ Lemma.

Another case that will be treated separately is that of $G = 2K_2$. Assume that $G$ is labelled as in Figure 4.13. Applying the $P_6$ Lemma to $s_4 r_2 v r_3 s_1$, we find a vertex $w$ that sees $s_4 s_1$ and misses $r_2 r_3 v$. Now $w$ must see $s_2$ and $s_3$ (to avoid a $C_5$ on $w s_1 s_2 r_2 s_4$ and $w s_4 s_3 r_3 s_1$, respectively), and $w$ must miss $r_1$ and $r_4$ (to avoid a $P_6$ on $w r_2 s_1 s_4 s_2 r_4$ and $w r_3 s_4 s_1 s_3 r_4$, respectively).

Now we may assume that $G$ has at least eight vertices and that $G \neq 2K_2$. Let $r_i, r_j$ be as in the Reduction Lemma, and let $s_i, s_j$ be their respective mates with respect to the partition $[R,S]$. Observe that $G_R$ has a component $R_0$ that includes neither $r_i$ nor $r_j$. Let $S_0$ be the corresponding component of $G_S$. Let $r_i$ be any vertex in $R_0$. If $R_0 = \{ r_i \}$ then let $s_i$ be the mate of $r_i$, else let $s_i$ be the mate of any neighbour of $r_i$ in $G_R$. Note that $s_i$ is in $S_0$ and misses $r_i$. If $r_i, r_j$ are adjacent then the subgraph of
Figures 4.14.1 and 4.14.2  (top and bottom)
$G$ induced by \{ri, rj, ri, sl, sj, si\} is as in Figure 4.14.1, else it is as in Figure 4.14.2.

If conclusion (a) of the Reduction Lemma holds then, by the induction hypothesis, we find vertices $w_i$ and $w_j$ non-adjacent to $v$ such that $w_i$ is universal on $S - s_i$ and null on $R - r_i$, and $w_j$ is universal on $S - s_j$ and null on $R - r_j$. In case $r_i, r_j$ are non-adjacent, case (1A) of the Little Local Lemma guarantees that one of $w_i, w_j$ is universal on \{si, sj, si\} (and therefore on $S'$), and null on \{ri, rj, ri\} (and therefore on $R'$). In case $r_i, r_j$ are adjacent, case (2A) of the Little Local Lemma guarantees that $w_i$ in universal on \{si, sj, si\} and null on \{ri, rj, ri\}.

If conclusion (b) of the Reduction Lemma holds, then by the induction hypothesis, we find a vertex $w$ non-adjacent to $v$ such that $w$ is universal on $S - s_j$ and null on $R - r_i$. But now, by the Little Local Lemma (apply cases (1B) and (2B) if $r_i, r_j$ are respectively non-adjacent and adjacent), $w$ is universal on \{si, sj, si\} and null on \{ri, rj, ri\}. ■
Figures 4.15A to 4.15D  (top to bottom)
The Second Extension Lemma. Let $G$ be a strong mirror subgraph of a graph $H$ in $M$, let $[R,S]$ be a strong mirror partition of $G$, let $r_t$ be isolated in $G_R$, and let $v$ be a vertex in $H - G$ that is universal on $R - r_t$ and null on $S - s_t$.

1. If $v$ sees both $r_t$ and $s_t$

   then some vertex $w$ misses $v$, sees both $r_t$ and $s_t$, is universal on $S - s_t$ and null on $R - r_t$.

2. If $v$ sees $r_t$ and misses $s_t$

   then some vertex $w$ misses $v$, is universal on $S$ and null on $R$.

3. If $v$ misses both $r_t$ and $s_t$

   then there are vertices $w, x, y$ such that the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in Figure 4.15A, and such that

   both $x$ and $y$ are universal on $S - s_t$ and null on $R - r_t$, and $w$ is universal on $R - r_t$ and universal on $S - s_t$.

4. If $v$ misses $r_t$ and sees $s_t$

   then there are vertices $w, x, y$ such that the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in one of Figures 4.15B, 4.15C, 4.15D and such that

   both $x$ and $y$ are universal on $S - s_t$ and null on $R - r_t$, and $w$ is universal on $R - r_t$ and universal on $S - s_t$.

Proof. In all four cases, we shall argue by induction on the number of vertices in $G$. If $G$ has precisely six vertices then, in each of the four cases, the desired conclusion follows from the corresponding case of the $C_6$ Lemma; see Figure 4.7. Now assume that $G$ has at least eight vertices. Note that (since $r_t$ is isolated in $G_R$) neither $G_R$ nor $\bar{G_R}$ is $2K_2$; let $r_{i}, r_{j}$ be as in the Reduction Lemma.

If case (a) of the Reduction Lemma applies,
set \( G' = G - \{ r_i, s_i \} \), \( R' = R - r_i \), \( S' = S - s_i \);
set \( G'' = G - \{ r_j, s_j \} \), \( R'' = R - r_j \), \( S'' = S - s_j \).

If case (b) applies,
set \( G' = G - \{ r_i, s_j \} \), \( R' = R - r_i \), \( S' = S - s_j \);
set \( G'' = G - \{ r_j, s_i \} \), \( R'' = R - r_j \), \( S'' = S - s_i \).

Proof of (1). By the induction hypothesis, there is a vertex \( w \) that misses \( v \), sees both \( r_i \) and \( s_i \), and is universal on \( S' - s_i \) and null on \( R' - r_i \). Since \( w \) is universal on \( \{ r_i, s_i \} \), the \textit{Zero-Two Lemma} guarantees that \( w \) is either universal on \( S - s_i \) and null on \( R - r_i \) or null on \( S - s_i \) and universal on \( R - r_i \). To exclude the latter alternative, we only need recall that \( w \) is universal on \( S' - s_i \).

Proof of (2). By the induction hypothesis, there is a vertex \( w' \) that misses \( v \), is universal on \( S' \) and null on \( R' \); there is also a vertex \( w'' \) that misses \( v \), is universal on \( S'' \) and null on \( R'' \). By the \textit{Little Local Lemma}, one of \( w', w'' \) has the properties required of \( w \).

Proof of (3). By the induction hypothesis, there are vertices \( w, x, y \) such that the subgraph induced by \( \{ r_i, s_i, v, w, x, y \} \) is as in Figure 4.15A, and such that \( x \) and \( y \) are both universal on \( S' - s_i \) and null on \( R' - r_i \), and \( w \) is universal on \( R' - r_i \) and null on \( S' - s_i \). Since \( w \) is universal on \( \{ r_i, s_i \} \), the \textit{Zero-Two Lemma} guarantees that \( w \) is either universal on \( R - r_i \) and null on \( S - s_i \) or universal on \( S - s_i \) and null on \( R - r_i \). To exclude the latter alternative, we only need recall that \( w \) is universal on \( R' - r_i \).

The same argument shows that \( y \) is universal on \( S - s_i \) and null on \( R - r_i \). Finally, since \( v \) and \( s_i \) are twins with respect to \( G \), and since \( x \) is universal on \( \{ r_i, v \} \), the same argument used once again shows that \( x \) is universal on \( S - s_i \) and null on \( R - r_i \).
Proof of (4). Let $F$ be the subgraph of $G$ induced by $\{r_t, s_t, u, x, z, y\}$. By the induction hypothesis, there are vertices $w, x, y$ such that $F$ is as in one of Figures 4.15B, 4.15C, 4.15D, and such that $x$ and $y$ are both universal on $S' - s_t$ and null on $R' - r_t$, and $w$ is universal on $R' - r_t$ and null on $S' - s_t$. There are three cases to consider.

Case B: the subgraph $F$ is as in Figure 4.15B.

Since $s_t$ and $u$ are twins with respect to $G$, and since $w$ is universal on $\{r_t, u\}$, by the Zero-Two Lemma it follows that $w$ is either universal on $R - r_t$ and null on $S - s_t$ or null on $R - r_t$ and universal on $S - s_t$; to exclude the latter alternative, note that $w$ is universal on $R' - r_t$. Since $x$ and $y$ are respectively null and universal on $\{r_t, v\}$, the same argument shows that both $x$ and $y$ are universal on $S - s_t$ and null on $R - r_t$.

Case C: the subgraph $F$ is as in Figure 4.15C.

Since $x, y, w$ are each either universal or null on $\{r_t, s_t\}$, the Zero-Two Lemma together with $x$ and $y$ being universal on $S' - s_t$ and $w$ being universal on $R' - r_t$ imply that $x$ and $y$ are both universal on $S - s_t$ and null on $R - r_t$, and $w$ is universal on $R - r_t$ and null on $S - s_t$.

Case D: the subgraph $F$ is as in Figure 4.15D.

Since $x, w$ are both universal on $\{r_t, s_t\}$, the Zero-Two Lemma together with $x$ being universal on $S' - s_t$ and $w$ being universal on $R' - r_t$ imply that $x$ is universal on $S - s_t$ and null on $R - r_t$, and $w$ is universal on $R - r_t$ and null on $S - s_t$. Finally, note that $u$ and $s_t$ are twins with respect to $G$, and that $y$ is universal on $\{r_t, v\}$. Now the Zero-Two Lemma together with $y$ being universal on $S' - s_t$ implies that $y$ is universal on $S - s_t$ and null on $R - r_t$. This concludes the proof of the Second Extension Lemma. \[\square\]
We now prove the main result of this chapter, namely, that the only unbreakable murky graphs are $L_8$, $L_9$, and strong mirror graphs.

**Proof of Theorem 4.1.** Let $H$ be an unbreakable murky graph. If $H$ contains $L_8$ as an induced subgraph, then by the $L_8$ Lemma, $H$ is either $L_8$ or $L_9$.

Thus we may assume that $H$ does not contain $L_8$, and so $H$ is in $\mathcal{M}$. Now note that the *WT Star Cutset Theorem* of Chapter 3 guarantees that $H$ contains a chordless cycle with at least five vertices, or the complement of such a cycle. Since $H$ is murky, $H$ does not contain $C_5$, $C_k$, or $\overline{C_k}$, for $k \geq 7$. Thus $H$ contains either $C_6$ or $\overline{C_6}$ as an induced subgraph; note that both $C_6$ and $\overline{C_6}$ are strong mirror graphs.

Now let $G$ be any strong mirror subgraph of $H$ with the greatest number of vertices. If $G = H$ then we are done, so assume that $G$ is a proper subgraph of $H$; we will show that this leads to a contradiction.

Since $H$ is unbreakable, there is some vertex $v$ in $H - G$ that is partial on $G$. By the *Complement Lemma*, by taking the complement if necessary, we may assume that $G_R$ is disconnected (note that $v$ is partial on $G$ if and only if $v$ is partial on $\overline{G}$ in $\overline{H}$). By the *Attachment Lemma*, there are two possible cases.

**Case (i): there is a strong mirror partition $[R, S]$ of $G$ such that $v$ is universal on $R$ and null on $S$.**

In this case, by the *First Extension Lemma*, there is a vertex $w$ that misses $v$, is null on $R$, and universal on $S$. Let $R' = R + w$ and $S' = S + v$. Now we claim that the partition $[R', S']$, whose couples are $\{w, v\}$ and all couples of $[R, S]$, is a strong mirror partition. To justify this claim, we need only show that $R'$ and $S'$ are strong $P_4$-free graphs, that $f(G_{R'}, w) = f(G_{S'}, v)$, and that for every couple $\{r_j, s_j\}$ of $[R, S]$,
\[ f(G_R, r_j) = f(G_R, r_j) \quad \text{and} \quad f(G_S, s_j) = f(G_S, s_j). \]

Since \( G_R \) is a disconnected strong \( P_4 \)-free graph, it has at least two components. \( G_{R'} \) is formed by adding the isolated vertex \( w \) to \( G_R \); thus \( G_{R'} \) is \( P_4 \)-free, and has at least three components; thus \( G_{R'} \) is strong. Since \( w \) is an isolated vertex in \( G_{R'} \),
\[ f(G_{R'}, w) = 0. \]
Similarly, \( G_S' \) is a strong \( P_4 \)-free graph, and \( f(G_S', v) = 0. \)

Finally, let \( r_j \) be any vertex of \( R \), and let \( X \) be the vertex set of the component of \( G_R \) containing \( r_j \). Note that \( X \) is also the vertex set of the component of \( G_{R'} \) containing \( r_j \). If \( |X| \geq 2 \), then \( X \) is a homogeneous set of both \( G_R \) and \( G_{R'} \), and
\[ f(G_R, r_j) = f(G_X, r_j) = f(G_{R'}, r_j), \]
by the Localization Lemma. On the other hand, if \( |X| = 1 \), then \( r_j \) is a singleton in both \( G_R \) and \( G_{R'} \), and \( f(G_R, r_j) = 0 = f(G_{R'}, r_j). \)
Similarly, \( f(G_S, s_j) = f(G_{S'}, s_j) \) for all \( s_j \) in \( S \). Thus the claim holds in this case, and \( [R', S'] \) is a strong mirror partition, contradicting the assumption that \( G \) was a largest strong mirror subgraph of \( H \).

**Case (ii): in every strong mirror partition \([R, S]\) of \( G \) there is a couple \([r_i, s_i]\) such that \( v \) is a twin of one of \( r_i, s_i \) with respect to \( G - \{r_i, s_i\} \).**

By the Isolation Lemma, there is a strong mirror partition of \( G \) such that \( r_i \) is a singleton in whichever of \( G_R \) or \( G_{R'} \) is disconnected. By the Complement Lemma, \( \overline{G} \) is also a strong mirror graph, with the same partition; \( v \) is partial on \( G \) in \( H \) if and only if \( v \) is partial on \( \overline{G} \) in \( \overline{H} \). Thus, by taking the complement of \( H \) if necessary, we may assume that \( r_i \) is isolated in \( G_R \). Now \( v \) is a twin of either \( r_i \) or \( s_i \) with respect \( G - \{r_i, s_i\} \); by swapping \( R \) and \( S \) if necessary, we may assume that \( v \) is a twin of \( s_i \).
Thus \( v \) is universal on \( R - r_i \) and null on \( S - s_i \). Now the Second Extension Lemma applies, and there are four subcases to consider. (See Figure 4.7.)
Subcase (1): $v$ sees both $r_t$ and $s_t$, and some vertex $w$ misses $v$, sees both $r_t$ and $s_t$, and is universal on $S - s_t$ and null on $R - r_t$.

Let $R' = R + w$ and $S' = S + v$. It is a routine exercise to show that the partition $[R', S']$, whose couples are $\{w, s_t\}, \{r_t, v\}$, and all couples of $[R - r_t, S - s_t]$, is a strong mirror partition.

Subcase (2): $v$ sees $r_t$ and misses $s_t$, and some vertex $w$ misses $v$, is universal on $S$, and null on $R$.

Let $R' = R + w$ and $S' = S + v$. It is a routine exercise to show that the partition $[R', S']$, whose couples are $\{w, v\}$, and all couples of $[R, S]$, is a strong mirror partition.

Subcase (3): $v$ misses both $r_t$ and $s_t$, and there are vertices $x, y$ such that the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in Figure 4.15A, and such that $x$ and $y$ are both universal on $S - s_t$ and null on $R - r_t$, and $w$ is universal on $R - r_t$ and null on $S - s_t$.

Let $R' = R + \{x, y\}$ and let $S' = S + \{v, w\}$. It is a routine exercise to show that the partition $[R', S']$, whose couples are $\{r_t, w\}, \{x, s_t\}$ and $\{y, v\}$ and all couples of $[R - r_t, S - s_t]$ is a strong mirror partition.

Subcase (4A): $v$ misses $r_t$ and sees $s_t$, and there are vertices $w, x, y$ such that the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in Figure 4.15B, Figure 4.15C, or Figure 4.15D, and such that $x$ and $y$ are both universal on $S - s_t$ and null on $R - r_t$, and $w$ is universal on $R - r_t$ and null on $S - s_t$.

Let $R' = R + \{x, y\}$ and let $S' = S + \{v, w\}$. If the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in Figure 4.15B, then the partition $[R', S']$ with couples $\{r_t, s_t\}, \{x, w\}, \{y, v\}$, and all couples of $[R - r_t, S - s_t]$ is a strong mirror partition; if the subgraph induced by $\{r_t, s_t, v, w, x, y\}$ is as shown in Figure 4.15C, then the
partition \([R',S']\) with couples \(\{r_i,v\}\), \(\{y,w\}\), \(\{z,s_i\}\) and all couples of \([R - r_i,S - s_i]\) is a strong mirror partition;

if the subgraph induced by \(\{r_i,s_i,v,w,x,y\}\) is as shown in Figure 4.15D, then the partition \([R',S']\) with couples \(\{r_i,w\}\), \(\{y,v\}\), \(\{z,s_i\}\) and all couples of \([R - r_i,S - s_i]\) is a strong mirror partition;

Thus, in all four subcases there exists in \(H\) a strong mirror subgraph with more vertices than \(G\); this contradiction completes the proof of Theorem 4.1. \(\blacksquare\)
4.4 A Characterization of Unbreakable Murky Graphs

In the previous section we showed that if a murky graph is unbreakable, then it must be $L_8$, $L_9$ or a strong mirror graph. In this section, we will prove the converse, namely, that $L_8$, $L_9$ and strong mirror graphs are murky and unbreakable. These two results combine to give the following characterization of unbreakable murky graphs.

Theorem 4.3. A graph is murky and unbreakable if and only if it is either $L_8$, $L_9$, or a strong mirror graph.

The necessary half (i.e. the "only if" part) of the theorem is Theorem 4.1; thus to prove Theorem 4.3, we need only prove the sufficiency half of the theorem. This half of the theorem is proved as the following four propositions.

Proposition 1. The graphs $L_8$ and $L_9$ are murky.

Proposition 2. Mirror graphs are murky.

Proposition 3. The graphs $L_8$ and $L_9$ are unbreakable.

Proposition 4. Strong mirror graphs are unbreakable.

Proof of Proposition 1. Since removing a vertex from $L_9$ corresponds to removing an edge from $K_{3,3}$, it follows that every eight-vertex induced subgraph of $L_9$ is $L_8$. Also, removing a vertex of degree four from $L_8$ leaves $L_7$; removing a vertex of degree three leaves $K_7$. Thus every seven-vertex subgraph of $L_8$ and $L_9$ is $L_7$ or $K_7$. It is a routine matter to verify that $L_7$ is murky; since the complement of a murky graph is murky, $K_7$ is murky. Thus both $L_8$ and $L_9$ are murky.

Proof of Proposition 2. Recall the Mirror Proposition of Section 4.2: every induced subgraph of a mirror graph has twins or anti-twins. Since neither $C_5$, $P_6$, nor $P_6$ have either twins or anti-twins, mirror graphs cannot have $C_5$, $P_6$, or $P_6$ as induced
subgraphs; thus mirror graphs are murky.

Recall that the neighbourhood $N(v)$ of a vertex $v$ in a graph $G$ is the set of all vertices of $G - v$ that see $v$, and that the non-neighbourhood $M(v)$ is the set of all vertices of $G - v$ that miss $v$. A pure star cutset of a graph $G$ is a set $S = v \cup N(v)$, for some vertex $v$ in $G$, such that $G - S$ is disconnected. The difference between a pure star cutset and a star cutset is that a pure star cutset consists of a vertex together with all of its neighbours, whereas a star cutset consists of a vertex together with any subset of its neighbours. We will call a graph $G$ with at least three vertices breakable if either $G$ or $\overline{G}$ has a star cutset. The following claim helps to shorten the proof of the final two propositions.

**Claim** (Chvátal, private communication). *Let $G$ be a breakable graph with at least five vertices. Then either $G$ or $\overline{G}$ has a pure star cutset.*

**Proof of Claim.** Let $G$ be a breakable graph with no pure star cutset. Chvátal observed [1985a] that this implies the existence of vertices $v, w$ in $G$, such that $v$ sees $w$, and $v$ dominates $w$. Now, if $v$ and $w$ have any common neighbour $x$ in $G$, then, in $\overline{G}$, $w \cup N(w)$ is a pure star cutset of $\overline{G}$. (In $\overline{G}$, removing $w$ and all its neighbours leaves a graph in which $v$ is a singleton, and $x$ is in some other component.) Thus we may assume that the only neighbour of $w$ in $G$ is $v$. Let $H = G - \{v, w\}$. Now there are two cases to consider.

**Case 1:** some vertex $z$ (other than $w$) sees $v$ and misses some $h \in H$.

In this case, we are done: in $G$, $z \cup N(z)$ is a pure star cutset.

**Case 2:** every vertex $z$ (other than $z$) that sees $v$ sees all vertices in $H$.

Let $S$ be the set of vertices of $H$ that see $v$, and let $T$ be all other vertices of $H$. Note that the hypothesis of Case 2 implies that $S$ is a clique, and that every vertex in $S$ sees
every vertex in $T$. Now, if there are any two non-adjacent vertices $a, b$ in $T$, then, in $G$, $a \cup N(a)$ (which includes all of $S$) is a pure star cutset. Otherwise, $T$ is a clique, and therefore the vertices of $H$ form a clique. But now, there is a vertex $h \in H$ such that in $\overline{G}$, $h \cup N(h)$ is a pure star cutset: if $T$ is non-empty, pick $h$ any vertex in $T$; else, pick $h$ any vertex in $S$ (in each case, in $\overline{G}$, $N(h)$ is a subset of $\{u, v\}$, and the vertices in $M(h)$ form a stable set; since $G$ has at least five vertices, the stable set has at least two.) This completes the justification of the Claim. 

**Proof of Proposition 3.** To prove that $L_8$ is unbreakable, by the preceding claim and the fact that $L_8$ is self-complementary, we need only prove that $L_8$ has no pure star cutset: we need only prove that, for each vertex $v \in L_8$, $M(v)$ is connected. 

An automorphism of a graph $G$ is a permutation $P$ of the vertices such that $x$ and $y$ are adjacent if and only if $P(x)$ and $P(y)$ are adjacent, for all pairs of vertices $x$ and $y$. Note that for every pair of vertices of $L_8$ with the same degree, there is an automorphism which maps one vertex to the other. Label the vertices of $L_8$ as in Figure 4.3. Vertex 1 has degree 4; the subgraph induced by $M(1)$ is a $P_3$, and is hence connected. Vertex 5 has degree 3; the subgraph induced by $M(5)$ is a $C_4$, and is hence connected. Thus $L_8$ is unbreakable.

To prove that $L_9$ is unbreakable, by the preceding claim and the fact that $L_9$ is self-complementary, we need only prove that $L_9$ has no pure star cutset; i.e. we need only prove that, for each vertex $v \in L_9$, $M(v)$ is connected. Note that for any two vertices in $L_9$ there is an automorphism which maps one vertex to the other. Thus we need only show that, for any vertex $v$ of $L_9$, $M(v)$ is connected. Pick any vertex of $L_9$; its non-neighbourhood induces a $C_4$, and is hence connected. Thus $L_9$ is unbreakable.
Proof of Proposition 4. To prove that a strong mirror graph is unbreakable, by the preceding claim and the fact that $\overline{G}$ is a strong mirror graph (see the Complement Lemma) we need only show that no vertex in $G$ has a pure star cutset. By the Isolation Lemma, there is a strong mirror partition $[R, S]$ such that ($v$ is in $R$ and) $v$ is a singleton in whichever of $G_R$ or $\overline{G}_R$ is disconnected. Let $w$ be the mate of $v$.

Case 1: $G_R$ is disconnected.

In this case $f(G_R, v) = 0$, and $v$ misses $w$. Thus $M(v) = R - v + w$; since $w$ sees all of $R - v$, $M(v)$ is connected.

Case 2: $\overline{G}_R$ is disconnected.

In this case $f(G_R, v) = 1$, and $v$ sees $w$. Thus $M(v) = R - v$. But since $G$ is strong, the fact that $v$ is a singleton in $\overline{G}_R$ implies that $\overline{G}_R$ has at least three components, and so $\overline{G}_{R-v}$ is disconnected, and therefore $G_{R-v}$ is connected. This concludes the proof of Proposition 4, and also the proof of Theorem 4.3. \[ \]
Appendix

The following result appears as Theorem 2.2.1 in Olariu [1988].

Theorem (Olariu). No minimal imperfect graph contains anti-twins.

Proof. Assume the statement false: some minimal imperfect graph $G$ contains anti-twins $u$ and $v$. Let $A$ denote the set of all neighbours of $u$ other than $v$, and let $B$ denote the set of neighbours of $v$ other than $u$.

Let $\alpha$ and $\omega$ denote the number of vertices in a largest stable set and clique respectively of $G$. Now

$B$ contains a clique of size $\omega - 1$ that extends

into no clique of size $\omega$ in $A \cup B$. \hfill (*)&

To justify (*), colour $G - v$ by $\omega$ colours and let $S$ be the colour class that includes $v$. Since $G - S$ cannot be coloured by $\omega - 1$ colours, it contains a clique of size $\omega$; since $G - S - v$ is coloured by $\omega - 1$ colours, it must be that $v \in C$. Hence $C - v$ is a clique in $B$ of size $\omega - 1$.

If a vertex $z$ extends $C - v$ into a clique of size $\omega$ then $z \notin B$ (since otherwise $z$ would extend $C$ into a clique of size $\omega + 1$). Thus (*) is justified.

The Perfect Graph Theorem guarantees that the complement of $G$ is minimal imperfect; thus (*) implies that

$A$ contains a stable set of size $\alpha - 1$ that extends

into no stable set of size $\alpha$ in $A \cup B$. \hfill (***)

Now let $C$ be the clique featured in (*) and let $S$ be the stable set featured in (**); let $X$ be a vertex in $C$ that has the smallest number of neighbours in $S$. By (**), $x$ has a neighbour $z$ in $S$; by (*), $z$ is non-adjacent to some $y$ in $C$. Since $y$ has at least as many neighbours in $S$ as $x$, it must have a neighbour $w$ in $S$ that is non-adjacent to $x$. Now $u, z, x, y, w$ induce in $G$ a chordless cycle. Thus $G$ is not minimal imperfect. \qed
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