MR3931302 91A46 00A08
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★Hex, inside and out—the full story.

CRC Press, Boca Raton, FL, 2019. *xxii*+297 *pp.*
ISBN 978-0-367-14422-7; 978-0-367-14425-8

The *full* story of the game of Hex as told in this ambitious book is in fact two rather distinct stories. On the one hand, there is a deeply human story of how the game of Hex came about and how it has developed over the years; this is a fascinating historical tale full of interesting and often brilliant characters. On the other hand, there is also what is essentially a mathematical story of the game itself; what mathematical theory is involved, what sort of strategies are needed to play the game well, how Hex is similar to or different from games such as chess and Go, and how well computers do playing against expert human Hex players.

The game of Hex was invented in 1942 by the Danish engineer, inventor, and poet, Piet Hein—who also invented the Soma cube and was described by Martin Gardner as "one of the most remarkable men in Denmark". His books of poetry were best sellers throughout Denmark. Here is a short poem published soon after Germany invaded Denmark in April 1940 in which he warns his fellow Danes, having just lost their "freedom", not to throw away their "honor" by collaborating with the Nazis:

"Losing one glove / is certainly painful, / but nothing / compared to the pain / of losing one, / throwing away the other, / and finding / the first one again."

Hex is played on a diamond-shaped board (a rhombus) made up of hexagonal cells. An 11×11 board is typical, but other sizes are also used. Smaller boards are good for beginners and for problems and puzzles. One pair of opposite sides is called black and the other pair white. As in Go, one player plays with black pieces and the other player with white pieces and they alternate play by placing one of their pieces on any unoccupied cell. The goal for each player is the same: connect the two opposite sides of the board having their color with a chain of pieces of that same color. The first player to do so wins.

Hein introduced his game, which he called Polygon, to the public on December 26, 1942, in an article in the leading Danish newspaper *Politiken*. To popularize the game he arranged to provide the newspaper with regular columns, while they produced and sold 50,000 50-sheet game pads—on each sheet was printed an 11×11 game board whose hexagonal cells could be marked by the players with either an asterisk (for black) or a circle (for white). The game became an immediate success. Sadly, because of the war Hein was soon forced to go underground and had to abandon his Polygon columns.

The story of the birth and early success of Hex has been meticulously researched here and is extremely well told. In particular, the book contains many wonderful old photographs: one from about 1930 showing Piet Hein as an eager-looking university student sitting behind his supervisor Niels Bohr and Werner Heisenberg, a picture of the front and back cover of a 50-sheet Polygon pad, a candid photo of the French graph theorist Claude Berge in 1974 playing a game of Hex hunched over a beautiful teak board with cigarette in hand, and a picture of a game between Hein and his wife on a game sheet. But there are also a good many pictures that don't contribute much to the story and should have just been left out of the book entirely, such as one showing a 1942 invoice for printing Polygon pads, or Hein's contract (in Danish) with *Politiken*,

or a Norwegian patent application for Polygon whose print is so tiny you can't even tell whether it is written in Danish or Norwegian.

While working on this book the authors made a major discovery. Hein did not write his amazing Polygon columns by himself; he had a partner, a university friend named Jens Lindhard who at 20 was among the best chess players in Denmark. Lindhard quickly became an expert player of Hein's new game and a partnership formed: Hein wrote the columns, but it was Lindhard who provided him with the exquisite Polygon puzzles that made the columns so enormously popular. Lindhard would go on to study physics with Niels Bohr and become a professor of physics at Aarhus University. It was there, twenty years after Lindhard's death, that the authors found a storage box containing a treasure trove of Polygon material: letters from Hein, diagrams for a winning strategy on a 6×6 board, *Politiken* contest entries, and puzzle drafts. It is truly thrilling to see one of these original puzzle drafts by Lindhard (though considerably less thrilling to also see a picture of him at university playing field hockey).

Hex was reinvented independently in 1948 by John Nash, then a beginning graduate student in mathematics at Princeton. Nash would later win a Nobel prize in economics in 1994 (for work he did as a graduate student in game theory) and an Abel prize in mathematics in 2015, and also achieve considerable fame as the subject of both Sylvia Nasar's biography *A beautiful mind* [Simon & Schuster, New York, 1998; MR1631630] and an Oscar-winning film of the same name starring Russell Crowe. In 1950 Parker Brothers marketed the game under the name Hex, but it was not at all successful and was soon discontinued. Things changed completely in July 1957 when Martin Gardner introduced Hex to the world in his *Scientific American* column.

Although Piet Hein and Jens Lindhard were aware of certain mathematical aspects of the game, the mathematical story of Hex really begins with Nash. By 1949 Nash had proven two fundamentally important properties of the game. He was the first to prove that a game of Hex cannot end in a draw, that is, any Hex board completely filled with black and white pieces must necessarily have either a chain of black pieces connecting the two black sides or a chain of white pieces connecting the two white sides—a highly non-trivial result equivalent to the Brouwer fixed-point theorem in topology. Nash also proved that on any $n \times n$ board the first player can always win. However, this is an existence proof (and does not provide a winning strategy).

The authors therefore give considerable attention throughout the book to first-player strategy. For example, in one diagram they show *all* winning first-player moves on any $n \times n$ board where $n \leq 5$. Contained in the box uncovered at Aarhus University were six diagrams by Lindhard which show that by 1943 he had found a winning first-player strategy for the 6×6 board: white, the first player, plays a white piece on either of the two center cells in the middle column; Lindhard splits all 35 responses by black into just six cases, and uses his six beautifully designed diagrams to illustrate white's winning play. The elegant puzzles provided by Lindhard for Hein's *Politiken* column followed a similar theme and, like in chess puzzles, ask the reader to find the winning move for white.

Not surprisingly, the analysis of optimal play in Hex was radically transformed by the use of computers. By 1995 all winning first moves in a 6×6 game had been found; by 2003 all winning first moves in a 7×7 game had been found; by 2009 all winning first moves in an 8×8 game had been found; and by 2012 all winning first moves in a 9×9 game had been found. In 2013 two winning first moves in a 10×10 game were discovered, but the authors believe that finding a winning first move on the 11×11 board is still decades away. Equally interesting has been the progress of computer versus human play. In 2015 two different computer programs each defeated an expert human player 2-0 (and in 2016 a computer program soundly defeated the European

champion in a Go match). At this point humans still seem to have an advantage over computers on 13×13 boards and larger, but as the authors say, "The story of Hex continues". *John J. Watkins*