Book review of "Hex: The Full Story"

Chris Sangwin *

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In *Hex: The Full Story*, Ryan B. Hayward and Bjarne Taft provide a comprehensive history of the game of Hex, up to and including an account of contemporary work. The book has three parts (i) a history of Hex; (ii) a discussion of the theory behind Hex play; and (iii) recent work including the development of computer opponents.

You may be forgiven for never having come across Hex yourself. Indeed, I discovered it by accident in the search for a mathematical engagement activity for school students, and in common with many other mathematicians immediately fell in love with the game. It could well be the best strategy board game for two players.

Hex is played on a board made up n rows and m columns of hexagonal cells arranged in a rhombus (normally $n = m$). There are two players who each own two opposite sides of the board. Counters can be placed in any unoccupied hexes. Players alternately place counters of their colour to occupy hexes. To win, connect your sides of the board with a continuous path made from counters of your colour. Note that normally counters cannot be moved, and that the four corner hexagons each belong to both adjacent sides.

Despite the simplicity of the rules, Hex is both very difficult to master and surprisingly fun to play. An example game is shown in Figure 1. In this game Red moved first, taking a very strong centre position. Blue was unable to block, but put up a spirited defence before Red finally won, connecting the two red sides. A 7×7 board is adequate for a couple of training games but very soon becomes too small, and 11×11 is the minimum size for a sensible game.

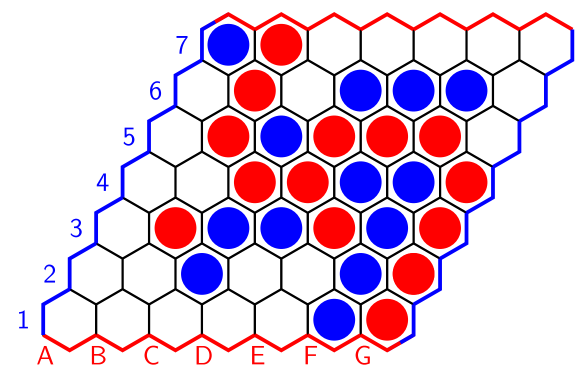

Figure 1: An example game of Hex

In Hex, it is best to play defensively: defence is also attack. By blocking your opponent you are also going to connect up your own sides. Also, since there is no hidden information or randomness, each player has *perfect knowledge*. A consequence is that if you can think of a strong response to your own move then look for a better one!

The game also makes use of patterns, known as templates. The simplest such template, shown in Figure 2, is called a *bridge*. These two red counters are strongly connected. If blue takes one empty hex then red immediately takes the other.

A further template is shown in Figure 3. The cells involved in the template are shaded in yellow, and

*The first author is a professor of mathematics education at The University of Edinburgh. His email address is C.J.Sangwin@ed.ac.uk.

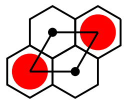

Figure 2: The bridge template

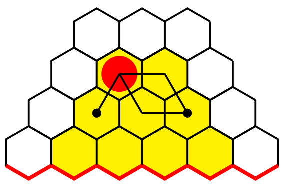

Figure 3: A template connecting Red to Red's edge

Red has occupied one cell of the template. If Blue plays in the shaded area, then provided Red responds immediately, Red can always connect to the red edge.

As players gain experience, the game evolves into a joust between patterns such as templates which ultimately must overlap. There is no need to always place your pieces in adjacent hexes. Use bridges to make connections between your pieces *and* simultaneously to block your opponent. But, beware of empty cells which are part of multiple overlapping templates.

There are two remarkable theorems. The first is that games can never end in a tie, i.e. one player must always win. The second is that the first player should win. For this reason, going first is possibly unfair. Normally pieces cannot be moved but one important exception used by most players is called the *swap rule*. On their first move the second player may move normally, or choose to swap their piece with that placed by the first player. This encourages the first player to only choose a moderately strong first move and so reduces any advantage of going first. For new players the swap rule is confusing and it is best to ignore the swap rule for the first few games. If you have not played Hex then now would be a good time to do so.

Originally the game was played on disposable pads, with each player marking their move in pencil. Nestor games (`https://nestorgames.com`) make a convenient roll-up set in 11×11 and also a 14×14 set, but there are few commercial alternatives of which I'm aware. Otherwise I have found Hex sets surprisingly difficult to purchase. Somewhat irked at this I made my own boards and a .pdf 11×11 board is available here `https://www.maths.ed.ac.uk/~csangwin/hex/` together with some basic notes. The computer opponent Hexy `http://vanshel.com/Hexy/` should be sufficient to thwart most beginners once their friends and relatives have become thoroughly frustrated.

Hex was invented by the Danish mathematician Piet Hein in the early 1940s, and he published a series of puzzles calling the game "Polygon" in the Danish newspaper Politiken. The same game was independently invented by John Nash in 1947 at Princeton University. Hex was popularized by Martin Gardner in his article [Gar59]. *Hex: The Full Story* contains a comprehensive history of the discovery and early development of the game, and this historical account is the heart of the book. The authors have provided extensive quotations from original letters, and large numbers of figures, showing original manuscript from Hein, Gardner and others involved in the early development of Hex. Photographs of the protagonists and background information complete this story. They sensitively explain the potential controversy between Hein and Nash about the discovery, and the key diplomatic role played by Martin Gardner in the whole affair. I confess I was absolutely fascinated by their account, and thoroughly recommend the book to anyone interested in Hex, or for scholars looking for source material on independent reinvention in science and mathematics.

For example, in notes for a lecture on 12 December 1942, (only rediscovered in 2004) Hein said

> What I intend to provide tonight is merely a sketch of a thought to an introduction to a game. The idea is to look at mathemat-

> ics as a game, and the game is a simple example of looking at games as mathematics. [HT19, p. 21]

This sentiment particularly interested me because my original motivation was to find a mathematical game as an engagement activity for school students. The idea of mathematics itself as a formal game is well established, but little did I know that this idea was shared by Hein in the context of Hex. I should add that my use of Hex was not entirely satisfactory: some young people got too consumed by the competitive element of the play and this robbed others of the pleasure of seeing the mathematical structure I hoped they would enjoy. Nevertheless I still think Hex can act as a model of mathematics itself, with formal rules, structure, patterns of play and heuristic thinking guiding strategy and thought.

The book also provides the story of the development of Hex Puzzles. Figure 4 shows four Hex puzzles taken from from [Gar59]. In each case Red is to move next and win. Right from the start, Hex was popularised by setting puzzles for the general public. It was natural to assume that Hein had authored these, but Chapter 4 suggests that Jens Lindhard had a leading role. Indeed the authors go as far as saying *"the Polygon project was a two-man partnership, with Hein the expositor and Lindhard the game expert"* [HT19, p. 65]. Lindhard's papers and archive, examined in 2017, contain puzzles and drafts for puzzles.

Hex is a connection strategy game, and there are many equivalent and related games. For example the "Game of Y" uses a triangular arrangement of hexagonal cells. Each player now has the same goal: connect all three sides with a continuous path of their colour. While Hex has players connecting opposite sides, in Y both players share the same goal. Boards can be made in a wide, bewildering even, variety of other ways, using polygons which are not necessarily all hexagonal. Underlying all these games is a connectivity graph, and viewed in this way such games can be recast as the *Shannon switching game*. Criteria on the connectivity graph are used to establish the two important theorems: (i) games do not end in a tie and (ii) first player to win. This book contains a very interesting account of early attempts to create automatic machines to play such games, including Shannon's machine to play Hex. The attempts to create automatic Hex opponents spans all epochs in the development of the computer, including analogue computers, digital computers and contemporary neural networks. Shannon's original device for playing Hex used current through resistors, effectively solving the discrete Hex problem by using an analogue machine.

Lastly, the authors discuss the theory of Hex play and give an account of how this has been used in very recent work to create computer opponents. The authors have their own significant part to play in this research, which is ongoing and tied to similar efforts with the game of Go. Linking history to contemporary mathematics is an unusual genre of writing, but I found it refreshing and welcome. The book itself is not long, since the generous use of illustrations and figures reduce the number of words.

I would not recommend this as a first book on how to play Hex. Instead, I'd recommend [Bro00], (and see also [Bro05] for comments on related games). I feel rather mean spirited writing this in a review. Instead this book is a highly valuable record of original source material, including a comprehensive reproduction of original Hex puzzles in the appendix which might be otherwise very difficult to obtain. I do think most Hex players will want a copy of this book, both for the puzzles and the historical background.

I'm saddened that so few people know about Hex, and that it has slipped out of consciousness. While Chess, Go and other classic games are well established in our cultural consciousness, Hex has yet to make this transition. So please, if you have not yet played Hex do so and then read this fascinating account of its invention, history and contemporary research to develop automatic opponents.

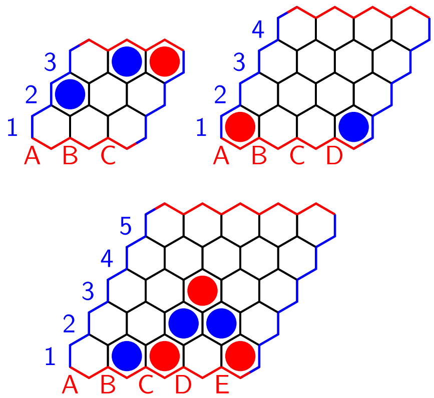

Figure 4: Hex puzzles: red to move and win

References

[Bro00] C. Browne, *Hex strategy: Making the right connections*, A K Peters/CRC Press, 2000.

[Bro05] ———, *Connection games: Variations on a theme*, A K Peters/CRC Press, 2005.

[Gar59] M. Gardner, *Mathematical puzzles and diversions*, Hammondsworth, 1959.

[HT19] R. B. Haywood and B. Toft, *Hex: The full story*, CRC Press, Raton, Florida, 2019.