sprague grundy theorem

warmup g = nim(1 3) canonical form?

move options? a nim(0 3), b nim(1 2), c nim(1 1), d nim(1 0)

game	canonical form	
*0 = 0	{ }	
*1 = { *0 } = *	{ *0 }	
*2	{ *0, *1 }	(exercise)
*t	{ *0, *1, ..., *t-1 }	(exercise)

a, b, c, d canonical forms?

```
d nim(1 0)        { *0 }        *1

a nim(0 3)        nim(3)        *3

c nim(1 1)        0             *0


b nim(1 2)   { nim(0 2), nim(1 1), nim(1 0) }

             = {   *2,       *0,       *1     }

             =               *3


   g = { nim(0 3), nim(1 2), nim(1 1), nim(1 0) }

     = {    *3,       *3,       *0,       *1    }


        simplify g?  use canonical form theorem
```

claim: for any non-zero non-equal nimbers *x, *y,   *x || *y

proof: *x < *y ?

iff *x + *y < *y + *y

iff *x + *y <   0   (tweedle-dee)

contradiction (Left Right indistinguishable,
only impartial outcome classes are P and N)

so to find can.form of impartial game, find
which options are reversible (you can forget about domination)

```
  g = nim(1 3)

    = { nim(0 3), nim(1 2), nim(1 1), nim(1 0) }

    = {    *3,        *3,        *0,        *1    }

  g can.form? find reversible options


claim: p2 (player 2) can reverse *3 to *2, because g = *2

proof: g = *2

  iff  g + *2  = *2 + *2

  iff  g + *2  = 0


       g + *2 = 0?   2 cases
```

```
p1 plays on g
   g *2 -> *3 *2 -> *2 *2 = 0 p2 wins
   g *2 -> *0 *2 -> *0 *0 = 0 p2 wins
   g *2 -> *1 *2 -> *1 *1 = 0 p2 wins
p1 plays on *2
   g *2 -> g *1 -> *1 *1 = 0 p2 wins
   g *2 -> g *0 -> *0 *0 = 0 p2 wins

so g + *2 = 0, so g = *2
```

by can.form.thm.II, we replace option *3 in G with
set of options from *2

(also need to confirm: neither *0, *1 are reversible in g,
this is left for you as an exercise)

```
so g = {     *3,       *0,  *1 }
     = {  *0,   *1,    *0,  *1 }
     = {  *0,   *1 }              <- can.form of *2
     = *2
```

so canonical form of g = nim(1,3) = *2 = is { *0, *1 } exercise find canonical form of g = nim(1 2 6)

g move options ?

(0 2 6) (1 0 6) (1 1 6) (1 2 0) (1 2 1) (1 2 2) (1 2 3) (1 2 4) (1 2

Lemma: let t be a normal-play impartial comb. game
with set S of move options X, such that every
move option is equal to some single-pile nim game,
i.e. S = {*t_j}, where each t_j is a non-negative integer.
Then t = *x, where x is the minimum excluded integer of {t_j}

```
example: assume t = {*1, *2, *6}
         mex{1,2,6} = 0, so t = *0

example: assume t = {*0, *1, *2, *4, *7}.
         mex(0,1,2,4,7} = 3, so t = *3.
```

Sprague-Grundy Theorem

for any normal-play impartial game g,

there is a non-negative integer m s.t. g = *m

proof: induction (use the lemma)

assume SGT holds for all smaller games

g = { g1, g2, ..., gt }

by induction, we can assume what?

g = { *n1, *n2, ..., *nt } where n1, n2, ..., nt are integers.

by lemma, g = *x, where x is mex{n1, n2, ..., nt} QED woo hoo :)

Sprague-Grundy theorem for nim

for any t-pile nim position g = (p1 p2 ... pt),

g = *x where x = nimsum(p1 p2 ... pt)

example nim(1 2 6)

= { nim(0 2 6) nim(1 0 6) nim(1 1 6) nim(1 2 0) ... nim(1 2 5) }

= { *4, *7, *6, *3, *2, *1, *0, *7, *6 }

= *5