cmput 304 2024 questions 7 (with hints) revised Nov 4

1. Assume node and arc lists are in alphabetic order. For each of these three digraphs:

   (a) draw the dfs trees and give the postorder sequence

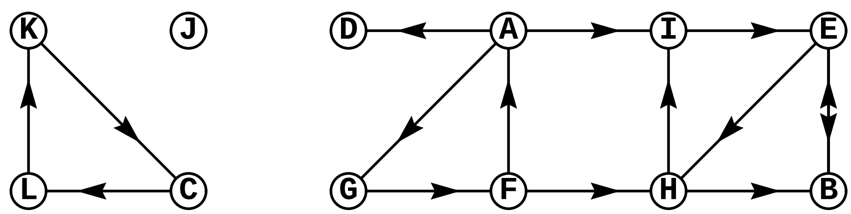
   (b) repeat for the transpose digraph

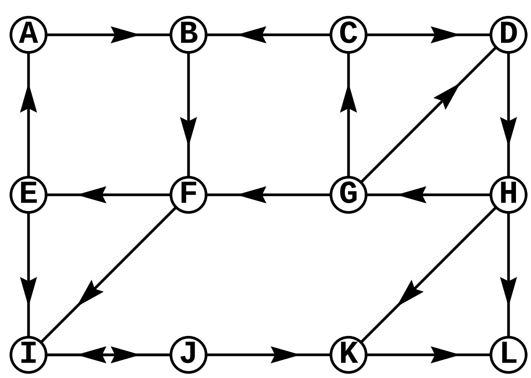
   (c) draw the dfs trees if node list is in reverse postorder of transpose

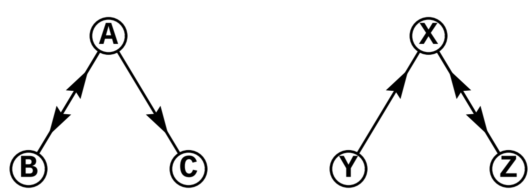
   (d) explain how to find digraph sccs from your answer to (c).

   Hint. Covered in the lectures. Check your answer using program `/graphs/decomp/scc24.py`

   .

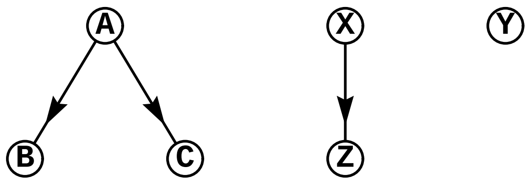






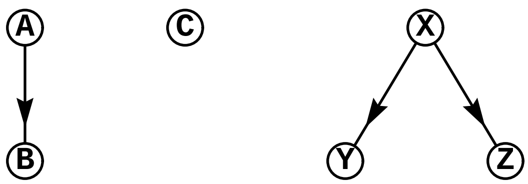
answer to previous question for bottom digraph

(a)



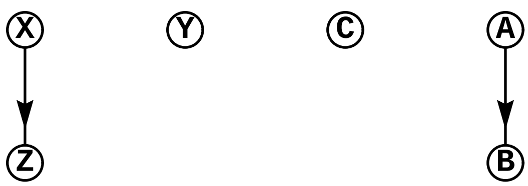
sequence `BCAZXY`

(b)



sequence `BACYZX`

(c)



(d) The sccs are exactly the trees of the final dfs,

e.g. for the bottom digraph, `{X,Z} {Y} {C} {A,B}` .

2. Prove: the sccs of a digraph are the sccs of its transpose.

   Hint: Consider two nodes x, y in the same scc of the digraph. Then there is a path from x to y and also from y to x in the digraph, and so also in the transpose (the second path gives the path from x to y in the transpose, the first give the path from y to x in the transpose). Thus x, y are also in the same scc of the transpose. The transpose of the transpose is the original digraph, so the preceding implies that if a, b are in the same scc of the transpose, then they are in the same scc of the original digraph. Thus the sccs of the two digraphs (the original and its transpose) are the same.

3. Let $T_1, T_2, \ldots, T_k$ be the trees, in the order they are created, from a dfs traversal of a digraph. Prove that there is no arc into T_k from any previous tree.

   Hint: Argue by contradiction: for some $j < k$, assume that there is some arc from x in T_j to y in T_k. Before the dfs from x finishes, the dfs from any node reachable from x must finish (or have previously finished). So the dfs from y must finish before the dfs from x, so y must be in T_z with $z \leq j$, contradicting the assumption that y is in T_k with $k > j$.

4. Prove: the last node z in the postorder sequence of a digraph is in a source scc.

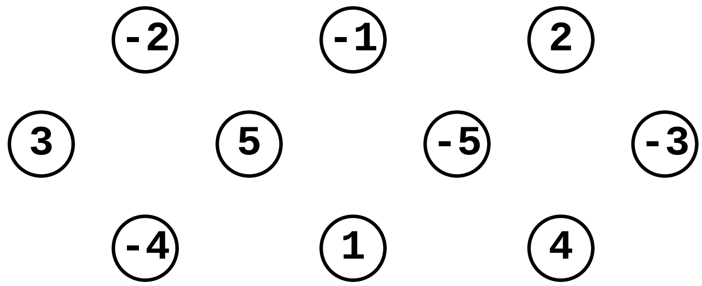
   Hint. Node z is the root of the last tree T_k of the dfs. If there is a node y such that there is a path from y to z, then by the previous question we know that y is also in T_k, so z and y are in the same scc. So there is no scc in the digraph that includes a node v with a path from v to z but no path from z to v. So z is in a source scc.

5. Prove: for a digraph D with transpose T, the first tree of a dfs of D that begins with the last node z in postorder of T is an scc of D.

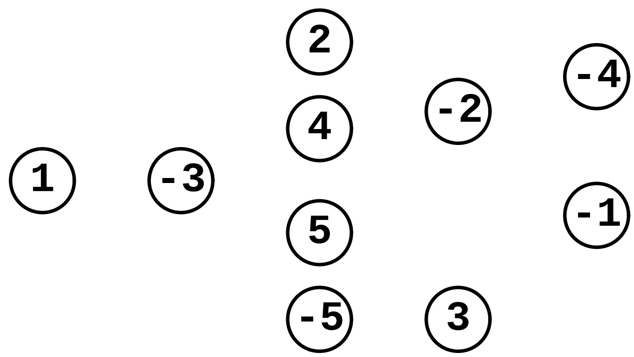
   Hint: z is in a source scc of T and thus a sink scc of D, so starting a dfs from z reaches only the nodes in this sink scc. So the set of nodes reachable from z, namely the nodes of the first tree of the dfs, is an scc.

6. (a) On the nodes below, draw the implication digraph for this 2-sat formula:

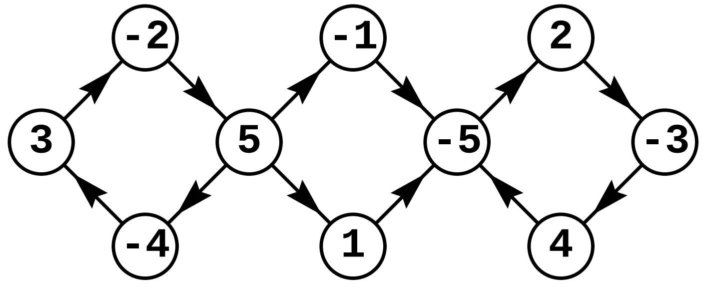
f = [1 -5] [-2 -3] [3 4] [-4 -5] [2 5] [-1 -5] .

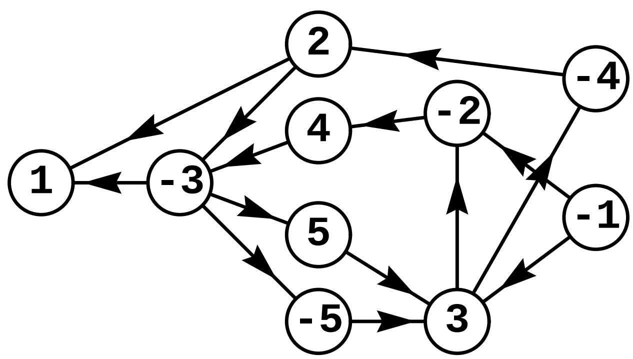


(b) Repeat for [1 -2] [1 3] [-2 -3] [2 4] [-3 -4] [3 -5] [3 5] .



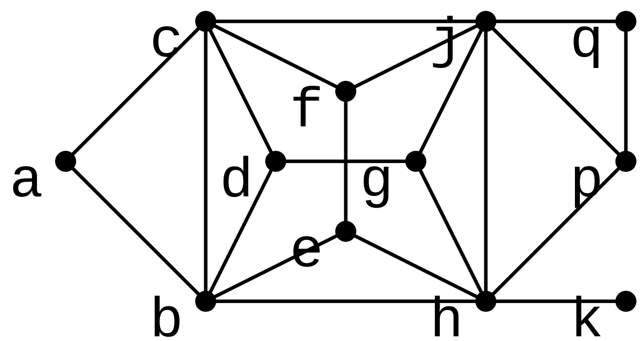
Hint. Answers on the next page.


-2
-1
2
3
5
-5
-3
-4
1
4


2
-4
4
-2
1
-3
-1
5
-5
3

Hints for all remaining questions at the end of this document.

7. Recall: for a graph $G = (V, E)$ and node subset $U \subseteq V$, $G[U]$ is the subgraph of G induced by U (all the nodes of U together with all edges of G that have both ends in U). For the graph G below, let $H = (X, F) = G[\{a, c, e, g, j, p, q\}$. Give X (the node set of H) and F (the edge set of H).



8. Recall: a *clique* is a graph $G = (V, E)$ is a subset K of V, such that for each pair of nodes x, y in K, (x, y) is in E. Recall: a clique in a graph is *maximal* if it is not a proper subset of any larger clique. For example, in the graph above $\{a, c\}$ is not-maximal clique and $\{a, b, c\}$ is a maximal clique.

   (a) List all maximal cliques of the above graph G.

   (b) List all maximal independent sets of $G[\{a, b, c, d, e, f, g, j, h\}]$.

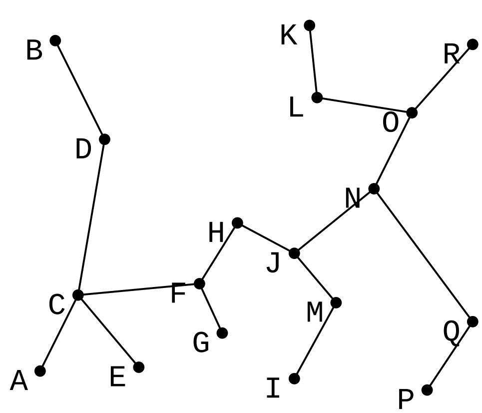
9. Recall: a node is *simplicial* in a graph if its set of neighbors is a clique.

   (a) List all the simplicial nodes in the graph in question 1.

   (b) Let v be simplicial node in a graph G. Let M be the set of non-neighbors of v. Let I_M be a largest independent set in $G[M]$. Prove that $\{v\} \cup I_M$ is an independent set in G.

   (c) Prove that $\{v\} \cup I_M$ is a largest independent set in G.

10. For the graph G below, find a largest independent set.



Hints

7. $X = \{a, c, e, g, j, p, q\}$. $F = \{(a, c), (c, j), (g, j), (j, p), (j, q), (p, q)\}$.

8. (a) You can check that there is no clique of size 4 in this graph. Thus each triangle (clique of size 3) is maximal. Also, you can check that every node is in at least one edge (clique of size 2) so there is no maximal clique of size 1. Also, you can check that there are exactly three edges that are not in any triangle, so these are maximal cliques of size two. So there are exactly 13 maximal cliques: the 10 triangles and the three edges $\{d, g\}$, $\{e, f\}$, $\{h, k\}$.

   (b) $\{a, d, e, j\}$, $\{a, d, f, h\}$, $\{a, d, e, g\}$, $\{a, e, j\}$, $\{a, f, h\}$.

9. (a) a, k, p

   (b) Rename $\{v\} \cup I_M$ as I'. Let x, y be any two nodes in I'. Then those two nodes are not adjacent. (Why? If one of the nodes is v, then the other is in the non-neighborhood of v and they are not adjacent. If neither node is v, then both are in I_M, which is an independent set, so they are not adjacent.)

   (c) Let I^+ be any largest independent set in G. Let I^+_M be the subgraph of I^+ restricted to the set M. Claim: I^+_M is a largest independent set of $G[M]$.

   Prove the claim by contradiction: assume that I^+_M is not a largest independent set of $G[M]$. Then there is some Z that is larger than I^+_M and an independent set of $G[M]$. Then $Z \cup \{v\}$ is an independent set of G (why? because v is non-adjacent with every node of M) and larger that $I^+ = I^+_M \cup \{v\} = I^+$, contradicting the assumption that I^+ is a largest independent set of G. So the claim holds.

   Now I^+ and $\{v\} \cup I_M$ are both independent sets and have the same size. I^+ is a maximum independent set of G, so $\{v\} \cup I_M$ is also a largest independent set of G.

10. There are many correct answers. Here is one: $\{A, B, E, G, H, I, K, N, P, R\}$.