cmput 304 2023 study questions 2

1. At right, unscramble these lines from `kruskalDemo.py`. Write line numbers only: indent properly. We have written the first line number for you.

```
a, b = t[0], t[1]                          #0  (3) ___ ___ ___ ___
if ra != rb:                               #1  ___ ___ ___ ___ ___
for v in G: P[v] = v                       #2  ___ ___ ___ ___ ___
L = createEdgeList(G)                      #3  ___ ___ ___ ___ ___
P = {}                                     #4  ___ ___ ___ ___ ___
print(a,b,t[2])                            #5  ___ ___ ___ ___ ___
ra, rb = UF.myfind(a,P), UF.myfind(b,P)    #6  ___ ___ ___ ___ ___
t = extractmin(L)                          #7  ___ ___ ___ ___ ___
UF.myunion(ra,rb,P)                        #8  ___ ___ ___ ___ ___
while len(L) > 0:                          #9  ___ ___ ___ ___ ___
```

2. Explain why the runtime of the extract-min implementation of Kruskal's algorithm is in $O(n^4 \lg n)$.

3. Define $T(n)$ as the total number of calls to `fib( )` made by the initial call `fib(n)`.

```
def fib(n):
  if (n<=1):
    return n
  return fib(n-1) + fib(n-2)
```

a) Explain why $T(0) = T(1) = 1$.

b) Explain why, for $n \geq 2$, $T(n) = 1 + T(n-1) + T(n-2)$.

c) Prove by induction on n that $T(n) = 2f(n+1) - 1$, where $f(n)$ is Fibonacci(n).

d) Using c, prove that $T(n)$ is in $\Theta(g^n)$, where g is the golden ratio.

4. The Fibonacci number $f(n)$ is defined as 0 if n is 0, 1 if n is 1, and $f(n-1)+f(n-2)$ for all integers $n \geq 2$. Prove by induction on j that, for all non-negative integers j, the value of a after line 4 has executed exactly j times is $f(j)$.

```
def ifib(n):                #line 0
  a,b = 0,1                 #line 1
  for _ in range(n):        #line 2
    print(a)                #line 3
    a,b = b, a+b            #line 4
  return a                  #line 5
```

5. a) Explain why any algorithm to add two n-bit numbers must read each of the $2n$ bits.

   b) Explain why the algorithm's runtime is in $\Omega(n)$.

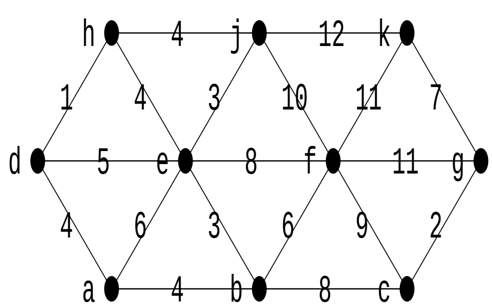
6. a) On the graph below, trace Prim's algorithm: starting from node `a`, list the edges as they are considered (consider edges with the same cost in alphabetic order, namely `ab, ae, ad, bc, be, bf, ...`) and then give the set of edges in the MST.

   b) Repeat a) starting from `k`.

   c) Repeat a) but consider same-cost edges in reverse-alphabetic order.

   d) Repeat b) but consider same-cost edges in reverse-alphabetic order.

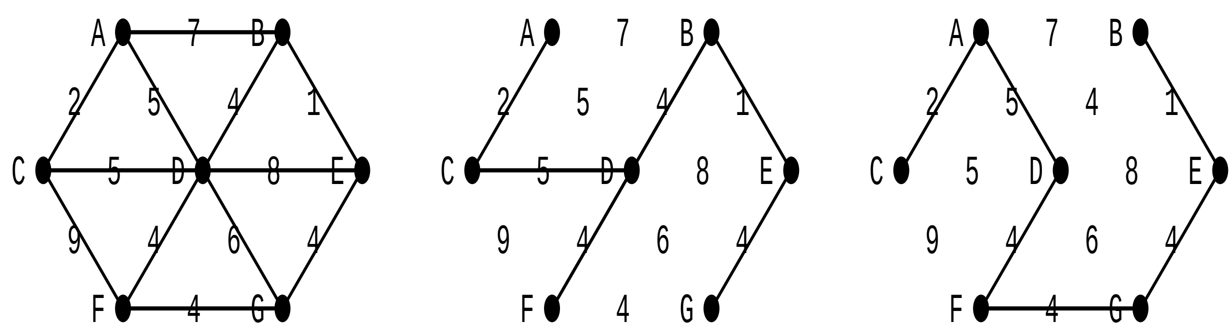
   e) Trace Kruskal's algorithm: list the edges of the MST in the order they are selected.



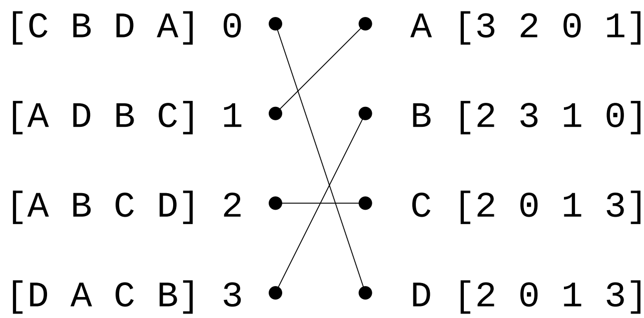
7. Here is a proof of correctness of Kruskal's algorithm. Justify (d), (k), (l), (m), (n), (o) and (p) finish the proof.

   (a) input: a graph G with n nodes, output: a set K of edges of G
   (b) we want to show that K is an MST
   (c) let M be any MST of G
   (d) case 1: if $K = M$ then we are done
   (e) case 2: assume K is not equal to M
   (f) label edges $e_1, e_2, \ldots, e_m$ in the order considered by the algorithm
   (g) let k be the smallest index such that e_k is not in M
   (h) let C be the cycle of $M + e_k$
   (i) let e_j be any edge of C that is not in K
   (j) let M_1 be $M + e_k - e_j$
   (k) M_1 is a spanning subgraph of G
   (l) M_1 is connected and so a spanning tree
   (m) cost(e_j) is not less than cost(e_k)
   (n) cost(e_j) is not greater than cost(e_j)
   (o) M_1 is an MST
   (p) ...rest of proof ...

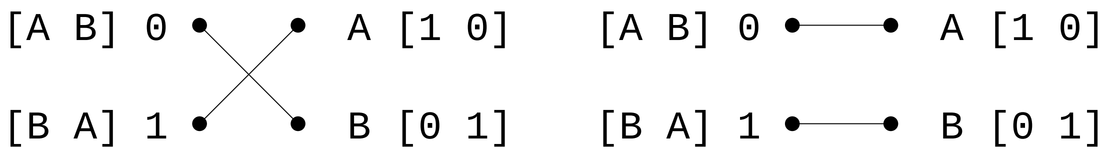
8. Below is a graph G, the edge set K returned by Kruskal's algorithm, and an MST M. In the above proof, what is e_k? What are the possible choices for e_j?



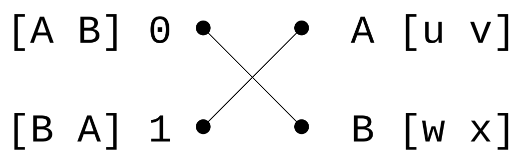
9. For the matching below, for each unmatched pair of nodes (h, r) with h in $\{0, 1, 2, 3\}$ and r in $\{A, B, C, D\}$, determine whether $\{h, r\}$ is an unhappy couple. Justify each answer.



10. For each bipartite preference system and matching below, is the matching stable? Justify.



11. a) Give all possible assignments of values to `u,v,w,x` below so the bipartite system is valid and the matching is stable. Hint: u,v = 1,0 or 0,1, same for w,x.



b) Repeat a) so that the system is valid and the matching is unstable.

12. Here is a diagram of the propose-maybe-reject stable matching algorithm after some number of rounds. a) For each proposer (hospital), in the next round, what new proposals are made? Explain carefully. b) After the new proposals are made, what rejections are made by maybe-rejecters (residents)? Explain carefully.