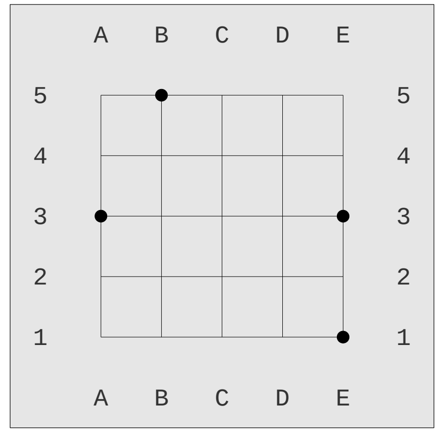
cmput 304 2024 study questions 1 show your work

recall: $2^{10} = 1024$ $\lg e = 1.4...$ $\lg 10 = 3.3...$

1. (a) Explain why any steiner tree for this puzzle must include at least 4 horizontal edges.
(b) Repeat (a) for vertical edges.
(c) Explain why a min cost solution to this puzzle must include at least 8 edges.
(d) Find a min cost solution to this puzzle.



2. A number q satisfies $\ln q = 529$. Give an arithmetic expression for $\lg q$ and estimate it roughly. (You do not have to calculate it). Repeat for $\log_{10} q$?

3. Let $x = 1,000,000$. Give an arithmetic expression for $\lg x$ and estimate it roughly. (You do not have to calculate it). Repeat for $\ln x$ and $\log_{10} x$.

4. Recall `http://webdocs.cs.ualberta.ca/~hayward/272/jem/collatz.html`. Prove or disprove (if you can):

   a) for all positive integers n, the last number printed by `collatz(n)` is 1.

   b) for all n in {1,2,...,10}, the last number printed by `collatz(n)` is 1.

   c) if the collatz conjecture fails for some integer, and if n_0 is the smallest such integer, then n_0 is odd.

5. Partition these functions into same-theta equivalence classes, in increasing order (so if $\lim_{n\to\infty} f(n)/g(n) = 0$ then put $f(n)$ before $g(n)$. Also, for each class, give the simplest function in that class. For example, the simplest function in the class $\Theta(3.5n^2 + 6\lg n)$ is $f(n) = n^2$.

   $a(n) = 2^n$ $b(n) = 3^n$ $c(n) = 1.5^{2n}$ $d(n) = n^2 + \lg n$ $e(n) = n^3 + (\lg n)^4$

   $f(n) = \ln n + 2\lg n + 16\log_{10} n$ $g(n) = n^{2.5} + (\lg n)^{90}$

   $h(n) = \ln n^2 + 0.5n^2 \lg n$ $k(n) = \ln n^2 + 0.5(n \lg n)^2$

6. Let $f(n) = n + \ln n$ and let $t(n) = n$. Find a positive rational number c and a positive integer n_0 such that, for all $n \geq n_0$, $f(n) \leq ct(n)$.

7. Let $f(n) = 2^n$. Let $g(n) = 1.7^n$. Using the defintion of $O(g(n))$, prove or disprove: $f(n) \in O(g(n))$.

8. Let $f(n) = n^2 + 100 \lg n$.

   (a) When $n = 2$, is $f(n)$ closer to n^2 or to $100 \lg n$?

   (b) Roughly what is the smallest n such that $n^2 \geq 100 \lg n$?

   (c) What is the simplest function $g(n)$ such that $f(n) \in \Theta(g(n))$?

9. a) For an input of size (number of bits) n, an algorithm has runtime in the set $\Theta(n^{2.5})$. For $n = 3.7e6$ (scientific notation: 3.7×10^6), the runtime is 11s. What is your best guess for the runtime when $n = 7.4e6$? Explain.

   b) Is it possible that your guess in a) will be 10 times too small? Explain.

10. For non-negative integers x, y with $x < y$, explain why the runtime of the usual addition algorithm is in $O(\lg y)$.

11. Repeat the previous question for time $\Omega(\lg y)$.

12. Recall iterative Fibonacci:

    `https://webdocs.cs.ualberta.ca/~hayward/304/jem/warmup.html#ifib`

    a) Justify each step in the runtime analysis:

    `https://webdocs.cs.ualberta.ca/~hayward/304/asn/itfib.pdf`

    b) Explain why the runtime of `ifib(n)` is proportional to $\sum_{j=1}^{n} \lg(\text{fib}(j))$.

    c) Explain why the above sum is in $O(n^2)$.

    d) Let $r(n)$ be the runtime for `ifib(n)`. For a positive integer t, explain why we expect that $r(2^{t+1})/r(2^t)$ will be around 4.

13. Recall recursive `fib(n)`:

```
def fib(n):
  if (n<=1): return n
  return fib(n-1) + fib(n-2)
```

Complete the proof of the following claim.

Claim. For all integers $n \geq 0$, `fib(n)` returns $f(n)$, where $f(n)$ is defined as 0 if n is 0, 1 if n is 1, and the sum of $f(n-1)$ and $f(n-2)$ when n is at least 2.

Proof. Argue by induction on n. The claim holds when n is 0 or 1 (why?).

Let x be an integer greater than or equal to 2. Assume that the claim holds for all values of n in the set $\{0, 1, \ldots, x-1\}$. In order to complete the proof, we now want to show that the claim holds when n is x, i.e. that `fib(x)` returns $f(x)$.

So what happens when `fib(x)` executes? Well, $x \geq 2$, so the `if` test evaluates to false, so the program returns `fib(x-1)+fib(x-2)`.

(now finish the proof ...)

Hints

- 1. (a) Any solution includes a path from the leftmost pin (in column A) to the rightmost pin (in column E) and so must travel a horizontal distance of 4 and so include at least 4 horizontal edges. (b) left for you (c) at least 4 horizontal and at least 4 vertical (d) left for you
- 9b. Consider the function $f(n) = n^{2.5} + (\ln n)^{90}$. See also the python3 program `thetatdemo.py` in the root directory of the class github demo:

  `https://github.com/ryanbhayward/algs`
- 12: covered in lecture Tuesday Sept 11.