cmput 304 2023 study questions 5 (with hints)

1. a) Describe a brute force algorithm that takes a graph as input and returns a largest clique.

   b) Describe a polytime algorithm that takes a graph and a node subset as input and reports whether the node subset is a clique.

   c) Explain how b) is used in proving that the k-clique problem is in the class NP.

2. *k-clique* is this problem:

   instance: a graph G and an integer k

   query: does G have a clique of size k?

   a) Define the problem *k-independent set*.

   b) Explain why the brute force method for solving k-clique runs in polytime if k is constant but in $\Omega(2^n/\sqrt{n})$ time when $k = n/2$.

   c) Give a polytime answer-preserving transformation T from k-clique to k-independent set (so T takes as input an instance of k-clique and gives as output an instance of k-independent set). Prove that T is polytime. Prove that T is answer-preserving.

3. a) Define NP-complete.

   b) So far in the lectures we have seen that these problems are NP-complete: conjunctive normal form sat (cnf-sat); 3-cnf-sat, also called 3-sat; k-clique; k-independent set.

   For each of these problems, explain briefly how we know that the problem is in the class NP-complete.

   c) Does there exist a polytime answer-preserving transformation from sat to k-independent set? If yes, explain briefly how you know this. If no, explain briefly why not.

   d) Is there a polytime answer-preserving transformation from k-independent set to sat? If yes, explain briefly how you know this. If no, explain briefly why not.

4. Let T be the transformation we saw in the lectures from cnf-sat to 3-sat. For the cnf-sat formula f represented below, give the corresponding 3-sat formula $T(f)$.

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f = [-1 3 5 6] [2 3 -4 -5 -6] [1 -2 3 -4 5 6]
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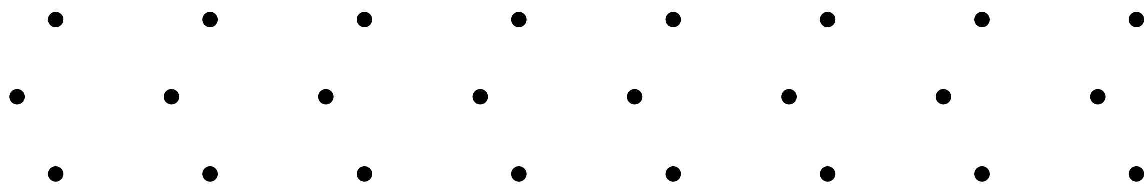
5. Let T be the transformation we saw in the lectures from 3-sat to k-independent set.

   a) For formula f below, label the nodes below and draw the edges of the graph $T(f)$.

$$\begin{aligned} f =& (\neg x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee \neg x_4) \wedge (x_1 \vee x_3 \vee x_4) \wedge (\neg x_1 \vee \neg x_2 \vee \neg x_4) \wedge \\ & (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_1 \vee x_2 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \end{aligned}$$



   b) If $T(f)$ has an 8-independent set, circle each node of such a set. If it does not, explain briefly.

   c) If f has a satisfying assignment, give it below. If it does not, explain briefly.

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variable     1  2  3  4
truth value __ __ __ __
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6. 2-coloring is this problem.

   instance: a graph G

   query: is there an assignment of at most 2 colors to nodes of G, such that pair of adjacent nodes have different colors?

   a) Give a graph with 3 nodes that is not 2-colorable, and a graph with 6 nodes and six edges that is 2-colorable.

   b) Give a polytime answer-preserving transformation T from 2-coloring to sat. Prove that T is polytime. Prove that T is answer-preserving.

   c) Does b) imply that 2-coloring is in the class NP-complete? Explain briefly.

7. Is there a polytime answer-preserving transformation from k-independent set to satisfiability? If yes, explain briefly how you know this. If no, explain briefly why not (or why this is unlikely to be true).

8. Let T be the transformation we saw in the lectures from 3-sat to k-independent set.

   a) For the 3-sat formula f below, draw the graph $T(f)$.

   $f = (x_1 \vee x_3 \vee \neg x_3) \wedge (\neg x_1 \vee \neg x_2 \vee x_4) \wedge (x_2 \vee x_3 \vee \neg x_4)$.

   b) Let f be any 3-sat formula with m clauses such that $T(f)$ has an independent set of size m. (i) Prove that $T(f)$ has no independent set of size $m+1$ (ii) Prove that f is satisfiable.

hints

1. a) for each vertex subset, check whether it is a clique. track largest-so-far b) for each pair of nodes in the subset, check whether they are adjacent. if any such pair is non-adjacent, it is not a clique c) for any yes-instance, there exists a k-clique, so find one (take as long as you want, say exponential time) and then use it for verification (can be checked in polytime)

2. a) replace the word *clique* with *independent set* b) the number of k-subsets of an n-set is n choose k, which is in $O(n^k)$, which is polynomial in n when k is constant, but in $\Omega(2^n/\sqrt{n})$ when $k = n/2$ (see the webnotes) c) given a graph G, replace it with its complement H (replace each adjacent pair of nodes with a non-adjacent pair, and vice versa).

3. a) see notes b) cnf-sat is NP-C by Cook's theorem. in class we saw that cnf-sat transforms to 3-sat, and that 3-sat transforms to k-clique, and in the previous question that k-clique transforms to k-independent set. c) yes: by b) we know that k-ind. set is NP-complete, so any problem in NP can be transformed into it, so sat (which is in NP) can be transformed into it (and here, everytime we say *transformed* we mean *answer-preserving polytime transformed*).

4. covered in the lectures

5. covered in the lectures

6. a) triangle. 6-cycle b) one boolean variable for each node. two colors, taupe (corresponding to value true) and fuscia (corresponding to value false). for each edge, say from node j to k, have the clause (v_j and not v_k) or (v_k and not v_j) c) no: the transformation is going in the wrong direction for that. we know from Cook's theorem that every problem in NP (e.g. 2-coloring) can be polytime transformed into sat, so seeing this transformation does not surprise us.

7. yes: every problem in NP (e.g. ind. set) transforms in any NP-complete problem (e.g. sat)

8. a) see the lecture bi) the graph partitions into m triangles. a triangle is a clique, so any independent set includes at most one node from each triangle, so any independent set has at most m nodes (otherwise, there would be some triangle that included at least two nodes from the IS, a contradiction) bii) see the lecture