cmput 304 2023 study questions 4

1. Consider the 0-1 knapsack problem. Below, show the output from the dynamic-program-by-weight (DPW) algorithm. Check your answer with `algs/hard/knap.py`.

```
val [6, 7,10, 7, 8]
wt  [4, 3, 6, 5, 3] W  15

   0   0   0   0   0   0   0
   1   0   0   0   0   0   0
   2   0   0   0   0   0   0
   3   0  __  __  __  __  __
   4   0  __  __  __  __  __
   5   0  __  __  __  __  __
   6   0  __  __  __  __  __
   7   0  __  __  __  __  __
   8   0  __  __  __  __  __
   9   0  __  __  __  __  __
  10   0  __  __  __  __  __
  11   0  __  __  __  __  __
  12   0  __  __  __  __  __
  13   0  __  __  __  __  __
  14   0  __  __  __  __  __
  15   0  __  __  __  __  __
```

2. Recall: a graph $G = (V, E)$ is defined by its node set V and edge set E. Recall: graphs $G = (V, E)$ and $H = (W, F)$ are *equal* if $V = W$ and $E = F$. Recall: graphs $G = (V, E)$ and $H = (W, F)$ are *isomorphic* if there is a bijection $f : V \leftrightarrow W$ such that $\{x, y\}$ is in E if and only if, for each edge $\{x, y\}$ in E, $\{f(x), f(y)\}$ is an edge in F.

   a) Let $G = (V, E)$ and $H = (W, F)$ with $V = \{1, 2, 3\}$, $W = \{a, b, c\}$, $E = \{\{1, 3\}, \{1, 2\}\}$, $F = \{\{a, c\}, \{b, c\}\}$. Prove/disprove: $G = H$.

   b) Continue from a): prove/disprove $G \cong H$.

3. Consider the 0-1 knapsack problem. Below, show the output from the dynamic-program-by-value (DPV) algorithm. Check your answer with `algs/hard/knap.py`.

```
val [3, 4, 6, 5, 5]
wt  [4, 3, 6, 5, 3]  W 18

  0   0   0   0   0   0
  1   -   -   -   -   -
  2   -   -   -   -   -
  3  __  __  __  __  __
  4  __  __  __  __  __
  5  __  __  __  __  __
  6  __  __  __  __  __
  7  __  __  __  __  __
  8  __  __  __  __  __
  9  __  __  __  __  __
 10  __  __  __  __  __
 11  __  __  __  __  __
 12  __  __  __  __  __
 13  __  __  __  __  __
 14  __  __  __  __  __
 15  __  __  __  __  __
 16  __  __  __  __  __
 17  __  __  __  __  __
 18  __  __  __  __  __
 19  __  __  __  __  __
 20  __  __  __  __  __
 21  __  __  __  __  __
 22  __  __  __  __  __
 23  __  __  __  __  __
```

4. Let $S(n)$ be $\{1, 2, \ldots, n\}$. Let $P(n)$ be the set of all subsets of $S(n)$. Let $f(n)$ the sum, over all p in P, of the size of p. E.g. $P(3) =$ `{ {}, {1}, {2}, {3}, {1,2}, {1,3}, {2,3}, {1,2,3} }`, so $f(3) = 0 + 1 + 1 + 1 + 2 + 2 + 2 + 3 = 12$. Prove by induction that $f(n) = n2^{n-1}$. Hint: create $S(n)$ by adding one item to $S(n-1)$. Hint: see the last page of this document.

5. Consider a 0-1 knapsack input with n items, each with a weight and value in {1, ..., 2n}. Consider the brute force (BF) algorithm.

   a) Explain why, over the whole algorithm, the number of the additions performed to sum the weight of each subset of items, is $n2^{n-1} - 2^n$. Hint: use the previous question.

   b) Explain why the worstcase total time to perform the additions in a) is in $\Theta(n2^n \lg n)$.

6. Let $a(n) = 1.01^n$, $b(n) = 2^{\sqrt{n}}$, and $c = \lim_{n\to\infty} a(n)/b(n)$. Does $c = 0$? Does $c = \infty$? Justify your answer. Hint: $k^n = (k^{\sqrt{n}})^{\sqrt{n}}$. Hint: last page.

7. Call a function $f(n)$ *superpolynomial* if, for any positive integer k,

   $\lim_{n\to\infty}(n^k)/f(n) = 0$,

   and call it *subexponential* if, for any positive rational ε,

   $\lim_{n\to\infty} f(n)/(1+\varepsilon)^n = 0$.

   Consider a 0-1 knapsack instance with n items, each weight and value n bits: for this instance, input size $f(n)$ is in $\Theta(n^2)$ and DPW worstcase runtime $t(n)$ is in $\Theta(n^4 2^n)$ (see `http://webdocs.cs.ualberta.ca/~hayward/204/jem/hard.html#knapdp`)

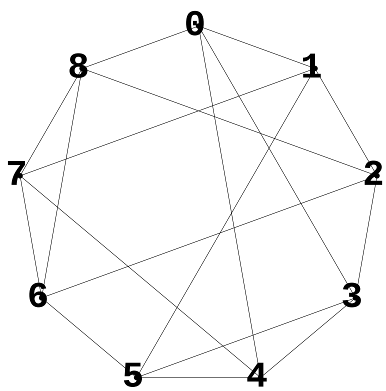
   a) Prove that $t(n)$ is superpolynomial.

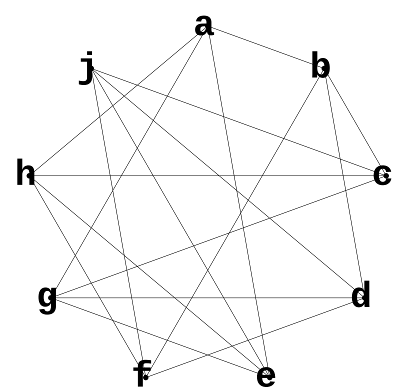
   b) Prove that $t(n)$ is subexponential.

8. Let G (left) and H (right) be the graphs below. a) Prove that this bijection $f : V \leftrightarrow W$ is not an isomorphism.

j	0	1	2	3	4	5	6	7	8
f(j)	a	b	c	d	e	f	g	h	j

   b) Continue from a): prove/disprove $G \cong H$.


0
1
2
3
4
5
6
7
8


a
b
c
d
e
f
g
h
j

9. *Graph isomorphism* is this problem:

   Input. Two graphs $G_0 = (V_0, E_0)$ and $G_1 = (V_1, E_1)$.

   Query. Is there an isomorphism $f : V_0 \to V_1$?

   Explain why graph isomorphism is in the class NP. Hint: is this a decision problem? If yes, is there a polytime verification algorithm for yes-instances?

10. Find a minimum set cover of these subsets:

```
     0 1 2 3 4 5 6 7 8 910111213141516171819
 S0  - - - - - * - - * - - - - - - - - - - -
 S1  - - - - - - - - - - - - - - * - - * - -
 S2  - - - - * - - - * * - * - * * - - - - *
 S3  - - - - - - - - - - * * - * - - - - - -
 S4  - - - - - - - * * - - - - * - - - - * -
 S5  - - * - - - - - - - * - - - - - - - - -
 S6  - * * - - - * - * - - * - - - * * - * -
 S7  - - - - - - - - - * - - - * * - - - * -
 S8  - - - * - - - - - - * - - - - * - * - -
 S9  * - - - - - - - - - - - * - - - - - - -
S10  - - - - - - * - - - - * - - * - - - - -
```

11. a) Using the format of question 9, give a careful definition of the set cover problem.

    b) Explain why set cover is in the class NP. Hint: is this a decision problem? If yes, is there a polytime verification algorithm for yes-instances?

some answers

2. Proof: $P(n)$ has 2^{n-1} subsets that contain item n and 2^{n-1} subsets that do not, so

$$
\begin{aligned}
f(n) = \quad & f(n-1) + 2^{n-1} \text{ from the subsets that contain item } n \\
& \qquad +f(n-1) \text{ from the subsets that do not} \\
= \quad & (n-1)2^{n-2} + 2^{n-1} + (n-1)2^{n-2} \\
= \quad & 2(n-1)2^{n-2} + 2^{n-1} \\
= \quad & (n-1)2^{n-1} + 2^{n-1} \\
= \quad & n2^{n-1}
\end{aligned}
$$

3. a) The number of additions for a subset of size t is $t-1$, so we subtract 1 once for each subset, so 2^n times, so the total number of additions is $n2^{n-1} - 2^n = 2^n(n-2)/2$.

   b) Each addition takes worst case $\Theta(\lg n)$ time: the maximum weight is 3n, which has about $\lg(3n) = \lg(3) + \lg(n)$ bits. So the total time is in $\Theta(n2^n \lg n) = \Theta(2^n n \lg n)$.

6.

$$
\begin{aligned}
\frac{1.01^n}{2^{\sqrt{n}}} &= \frac{(1.01^{\sqrt{n}})^{\sqrt{n}}}{(2)^{\sqrt{n}}} \\
&= \left(\frac{1.01^{\sqrt{n}}}{2}\right)^{\sqrt{n}}
\end{aligned}
$$

and since $\lim_{n\to\infty} 1.01^{\sqrt{n}}/2 = \infty$, $\lim_{n\to\infty} a(n)/b(n) = \infty$.