Quiz 5 Solutions 304 Fall 2022

1. [**1 marks**] Let s be the sum of the last 4 digits of your student id number. Let t be the sum of the digits of s. Let your **quiz code** u be t (mod 3). E.g. if your student id is 1234567, then s is $4+5+6+7=22$, t is $2+2=4$, and u is 4 (mod 3) $=1$.

   write your s: __________ write your t: __________ write your u: __________

2. [$2+3$ **marks**] If your quiz code u is respectively 0, 1, 2, use matrix M0, M1, M2 for this question. Consider the 2-player 0-sum matrix game with your matrix.

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M0  3 -1  0        M1  3  0 -1        M2  0  1 -4
    0  3 -4           -1  2  0            3 -3  0
   -2  0  1            0 -3  4           -2  0  1
```

   a) Assume that the row player plays the mixed strategy (1/4, 1/4, 1/2).

   What is the row player's expected score against column 1? __________

   against column 2? __________ against column 3? __________

   What is the column player's best choice? (1, 2, or 3) __________

   b) The row player wants to find her minimax mixed strategy. Formulate this problem as an LP, and then explain your answer.

Q1 Solution:

Question 1 is straightforward. Take for example your student ID to be 126363 and so s = 18. Then t = 9 and u = 0.

Q2.a Solution:

We choose matrix M0 based on our u. On any given round of the game, there is an $x_i y_j$ chance that Row and Column will play i th and j th moves, respectively. Therefore the expected (average) payoff is

$$\sum_{i,j} G_{ij} \cdot \text{ Prob[Row plays } i\text{, Column plays } j\Big] = \sum_{i,j} G_{ij} x_i y_j.$$

So the row player's expected score against column 1 would just be $\frac{1}{4}\cdot 3+\frac{1}{4}\cdot 0+\frac{1}{2}\cdot -2 = \frac{-1}{4}$.

Similarly So the row player's expected score against column 2 would just be $\frac{1}{4}\cdot -1+\frac{1}{4}\cdot 3+\frac{1}{2}\cdot 0 = \frac{1}{2}$.

Lastly, the row player's expected score against column 3 would just be $\frac{1}{4}\cdot 0+\frac{1}{4}\cdot -4+\frac{1}{2}\cdot 1 = \frac{-1}{2}$.

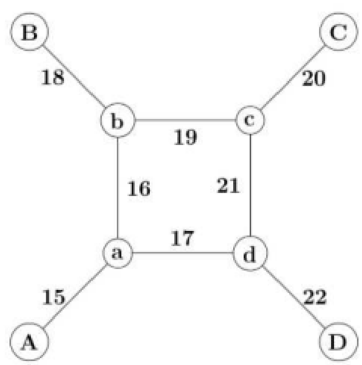
Clearly the column player's best choice would be choice 3.

Q2.b Solution:

Now if Row wants to find her minimax mixed strategy she knows that her best response will achieve an expected payoff of $\min\{3x_1 - 2x_3, -x_1 + 3x_2, -4x_2 + x_3\}$. She should choose $\mathbf{x}$ defensively to maximize her payoff against this best response: Pick (x_1, x_2, x_3) that maximizes $\underbrace{\min\{3x_1 - 2x_3, -x_1 + 3x_2, -4x_2 + x_3\}}_{\text{payoff from Column's best response to } \mathbf{x}}$ This choice of x_i 's gives Row the best possible guarantee about her expected payoff. And we will now see that it can be found by an LP! The LP is as follows

$$\begin{aligned} \max \quad & z \\ -3x_1 + 2x_3 + z &\leq 0 \\ x_1 - 3x_2 + z &\leq 0 \\ 4x_2 - x_3 + z &\leq 0 \\ x_1 + x_2 + x_3 &= 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

2. [1 + 4 + 1 **marks**] You manage a communications network with users A,B,C and bandwidths shown in the figure below (D is no longer involved in this project). You need to establish a connection between A-B, A-C, and B-C. These connections pay 5, 4, 3, dollars respectively per unit bandwidth. Between each pair of users at least 7 units must be routed.



Each connection has two possible routes. For A-B, xAB is traffic volume along A-a-b-B, yAB is volume along A-a-d-c-b-B; define xBC, yBC, similarly; xAC, yAC is traffic along A-a-b-c-C, A-a-d-c-D respectively. You want to maximize this network's revenue. Using the variables above, formulate this problem as an LP:

Q2.a b Solution:
This is a linear program. We have variables for each connection and each path (long or hort); for example, x_{AB} is the short-path bandwidth allocated to the connection between A nd B, and y_{AB} the long-path bandwidth for this same connection. The objective function and inequalities clearly would be after accounting for all the edge connections would be

$$
\begin{aligned}
&\max 5x_{AB} + 5y_{AB} + 3x_{BC} + 3y_{BC} + 4x_{AC} + 4y_{AC} \\
&x_{AB} + y_{AB} + x_{BC} + y_{BC} \leq 18 \quad \text{[edge } (b, B) \text{]} \\
&x_{AB} + y_{AB} + x_{AC} + y_{AC} \leq 15 \quad \text{[edge } (a, A)\text{]} \\
&x_{BC} + y_{BC} + x_{AC} + y_{AC} \leq 20 \quad \text{[edge } (c, C) \text{]} \\
&x_{AB} + y_{BC} + x_{AC} \leq 16 \quad \text{[edge } (a, b)\text{]} \\
&y_{AB} + x_{BC} + x_{AC} \leq 19 \quad \text{[edge } (b, c)\text{]} \\
&y_{AB} + y_{BC} + y_{AC} \leq 17 \quad \text{[edge } (a, d)\text{]} \\
&y_{AB} + y_{BC} + y_{AC} \leq 21 \quad \text{[edge } (c, d)\text{]} \\
&x_{AB} + y_{AB} \geq 7 \\
&x_{BC} + y_{BC} \geq 7 \\
&x_{AC} + y_{AC} \geq 7 \\
&x_{AB}, y_{AB}, x_{BC}, y_{BC}, y_{AC}, y_{AC} \geq 0
\end{aligned}
$$

Q2.c Solution:
Therefore a feasible solution would just be to set $x_{AB} = 7, x_{BC} = 7, x_{AC} = 6$ and $y_{AB} = 0, y_{BC} = 0, y_{AC} = 1$. This clearly satisfies all the constraints and the objective function value would just be now 84.