Quiz 1 Solutions 304 Fall 2022

1. Interpret the 4th and 5th digits of your student number as a 2-digit decimal number n. E.g. if your student number is ***70**, then n is 70. If n is square — 0, 1, 4, 9, 16, 25, 36, 49, 64, 81 — then increment n by 5. Now multiply n by 10^6. E.g. if your student number is ***70**, then n is now 70 00 00 00; if your student number is ***36**, then n is now 41 00 00 00.

   By hand, using the algorithm from class, find the largest integer t such that $t \leq \sqrt{n}$. Show your work.

   Solution: Taking the case where n is 41177.
   First let's break this number such that we have : $\overline{4}\ \ \overline{11}\ \ \overline{77}$
   Step 1: Let's find x such that $x * x \leq 4$, i.e x = 2
   Now, dividend is 4 and divisor is x.
   $\implies x = 2$ is the leftmost digit of our quotient.
   Remainder is 0 we now carry the next two digits 11.
   Step 2: Let's find y such that $(2 * 2 * 10 + y) * y \leq 11$, i.e y = 0
   Now, dividend is 11 and divisor is 4y i.e 40.
   $\implies y = 0$ is the second leftmost digit of our quotient
   Remainder is 11 , we now carry the next two digit 77.
   Step 3: Let's find z such that $(2 * 20 * 10 + z) * z \leq 1177$, i.e z = 2
   Now, dividend is 1177 and divisor is 402. $\implies z = 2$ is the third leftmost digit of our quotient
   Now after this the remainder is 373.

   So we got xyz as the decimal number which is 202 as the largest $t \leq \sqrt{41177}$.

2. Let $f(n) = 2^n$. Let $g(n) = 1.7^n$. Prove or disprove: $f(n) \in O(g(n))$.

solution: We prove $f(n) \notin g(n)$ by way of contradiction. Suppose $f(n) \in O(g(n))$ so by definition there is a constants $c > 0$, and an integer n_0 such that $f(n) \leq c \cdot g(n)$ for all $n \geq n_0$.

At this point we can argue in number of ways.

First argument:

$$\begin{aligned} 2^n \leq c \cdot 1.7^n &\iff (\frac{2}{1.7})^n \leq c \\ &\iff n \cdot \log_c(\frac{2}{1.7}) \leq \log_c(c) = 1 \\ &\iff n \leq \frac{1}{\log_c(\frac{2}{1.7})}, \end{aligned}$$

where the last line follows because $\frac{2}{1.7} > 1$ so $\log_c(\frac{2}{1.7}) > 0$ and dividing an inequality by a positive number preserves the inequality.

So for all $n > \frac{1}{\log_c(\frac{2}{1.7})}$, $f(n) > c \cdot g(n)$, a contradiction.

Second argument: $f(n) \leq c \cdot g(n)$ for $n \geq n_0$ so $\frac{f(n)}{g(n)} \leq c$ for $n \geq n_0$ so $\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq c$ however $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{2^n}{1.7^n} = \lim_{n\to\infty} (\frac{2}{1.7})^n = \infty$ where the last equality follows since $\frac{2}{1.7} > 0$, a contradiction.

Remarks: the $g(n)$ might have been different for you in the quiz like $g(n) = 1.6^n$ or $g(n) = 1.8^n$ but the solutions are similar.

3. Define $f(n)$ as 0 if n is 0, 1 if n is 1, and $f(n-1)+f(n-2)$ if n is an integer greater than or equal to 2. Consider this python procedure:

```
def fib(n):
  if (n<=1):
    return n
  return fib(n-1) + fib(n-2)
```

Claim. For all integers $n \geq 0$, `fib(n)` returns $f(n)$.

Proof. Argue by induction on n. The case when n is 0 or 1 was proved in the lecture.

Fix n to be some integer t greater than or equal to 2. Assume that the claim holds for all values of n in the set $\{0, 1, \ldots, t-1\}$. In order to complete the proof, we now want to show that the claim holds when n is t, i.e. that `fib(t)` returns $f(t)$.

So what happens when `fib(t)` executes? Well, $t \geq 2$, so the if condition evaluates to false, so the program returns `fib(t-1)+fib(t-2)`.

(now you complete the proof ...)

Solution: But we already know that `fib(t-1)+fib(t-2) =` $f(t-1) + f(t-2)$, since `fib(t-1) =` $f(t-1)$ and `fib(t-2) =` $f(t-2)$. However, $f(t-1)+f(t-2) = f(t)$ based on the definition of $f(n)$. Hence, `fib(t) =` $f(t)$, and the proof is complete.