# The Economist's View of Combinatorial Games

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# Overview

- Combinatorial Game Theory
- Goal and Motivation
- Economic Rules and Sentestrat
- Komaster and Extending Thermographs
- Approximations and Forecasting

# Combinatorial Game Theory (CGT)

Game assumptions:

- Two players (Left and Right) alternate turns
- Finite
- No elements of chance
- No hidden information
- No draws, ties
- Normal play: last player with legal move wins

# CGT Motivation

- Determining the value of a game under optimal play
- Determining the value of different moves
- Solving the sum of independent subgames

# Game Notation

Game $G = \{G^L | G^R\}$

- Game defined by Left and Right options
- Positive values favour Left, negative favour Right
- Zero is second-player win, fuzzy is first-player win

# Extension of Numbers

Numeric games: $G^L < G < G^R$

- $0 = \{|\}$, any second-player win is equivalent
- $1 = \{0|\}, \ldots, n = \{n-1|\}$
- $-G = \{-G^R| - G^L\}$, reverse roles
- $\{0|1\} = 1/2, \{0|1/2\} = 1/4, \ldots$

# Non-Numeric Values

- $* = \{0|0\}$, simplest fuzzy game
- $\uparrow = \{0|*\}$, $\downarrow = -\uparrow$
- $\pm x = \{x|-x\}$ (switches)

# Further Concepts

Sums of Independent Subgames:

$G = \{G^L | G^R\}, H = \{H^L | H^R\} \Rightarrow$

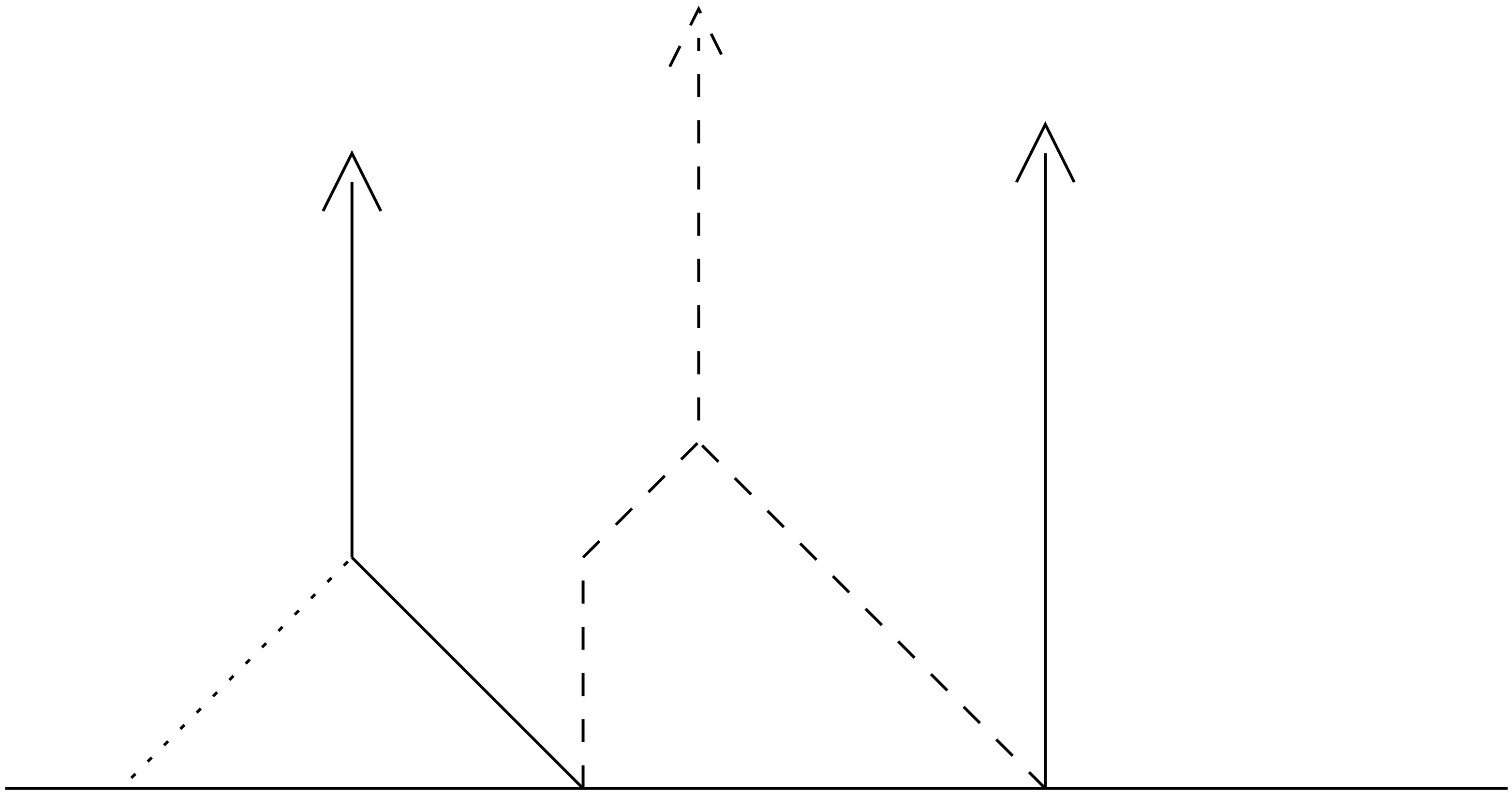
$G + H = \{G^L + H, G + H^L | G^R + H, G + H^R\}$

Incentives:

$\Delta^L(G) = G^L - G$

$\Delta^R(G) = G - G^R$

# Thermographs



# Motivation for New Approach

Goal: Extend computation of thermographs to handle positions containing kos

- Game positions no longer form a DAG
- Will not deal with interdependent kos (i.e. triple ko)

# Economist's Rules

Initial Setup:

- Bid for roles in each subgame
- Bid for privilege of first move
- Taxation - payment to opponent for making move

Player's Move:

- Pay and Play, or
- Pass

# Economist's Rules

Game Transitions:

- Both pass $\Rightarrow$ New bid on taxation (must decrease)
- Higher bid plays next, alternate if tie
- Game ends when neither willing to bid $\geq t_{min}$
- Zero-sum game, wealth determines victory

# Sentestrat

Bidding:

- Use mean for Left player bid
- Use temperature for tax/first move bid

Playing:

- If opponent increases temp, respond locally
- If temp $\geq$ tax, play in subgame of max temp
- If temp $<$ tax, pass

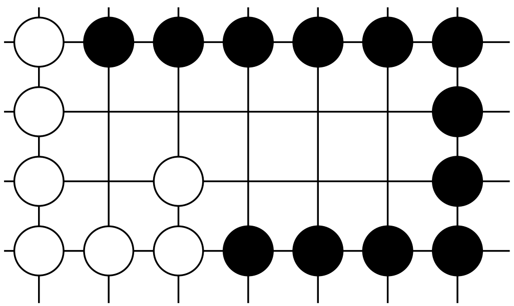
# Sentestrat Analysis

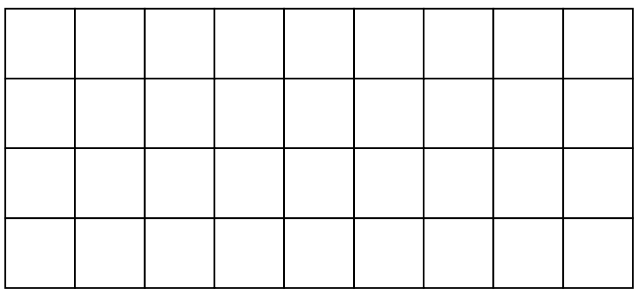
- If both players use Sentestrat, then bids always tie and play alternates
- Theorem: Sentestrat is optimal under Economist's Rules

# Subzero Temperatures

- $t_{min} = -1$, opponent pays you to play
- Mathematized go rules: groups become immortal, fill-in own territory

# Sample Game





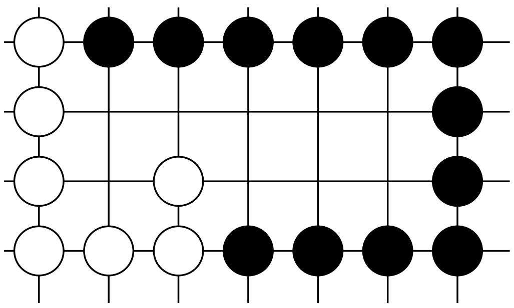
Initial Bids:

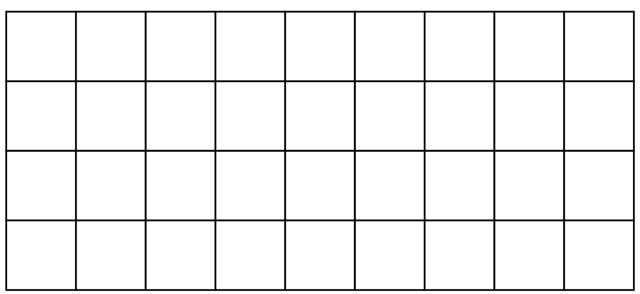
Player 1: pays 2 to play Black in Go

Player 1: pays 1 to play Vertical in Domineering

Player 2: pays 3/2 to play first

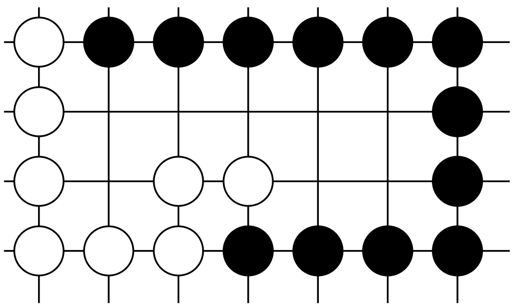
# Sample Game



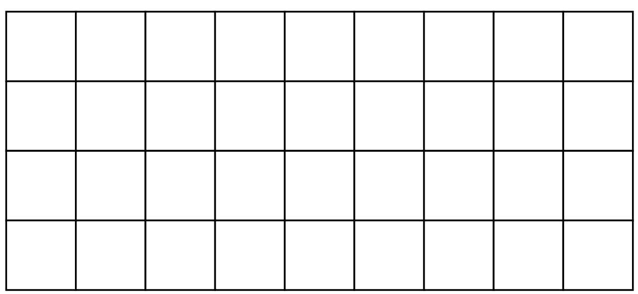


| Game | Mean | Temp |
|---|---|---|
| Go | 2 1/32 | 1 15/32 |
| Dom | 3/4 | 1 1/8 |

# Sample Game



| Game | Mean | Temp |
|---|---|---|
| Go | 9/16 | 1 1/16 |
| Dom | 3/4 | 1 1/8 |



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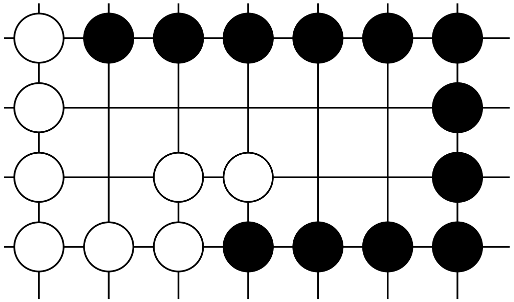
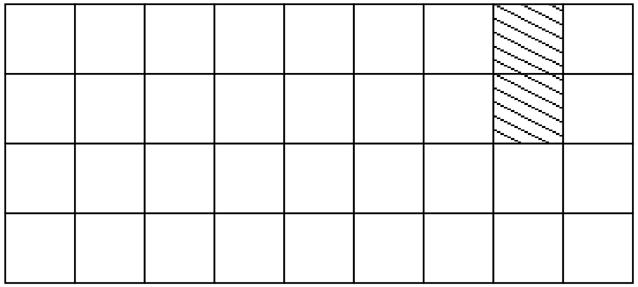


| Game | Mean | Temp |
|---|---|---|
| Go | 9/16 | 1 1/16 |
| Dom | 1 7/8 | 1 1/8 |



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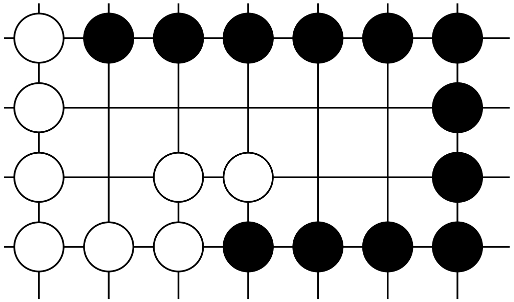
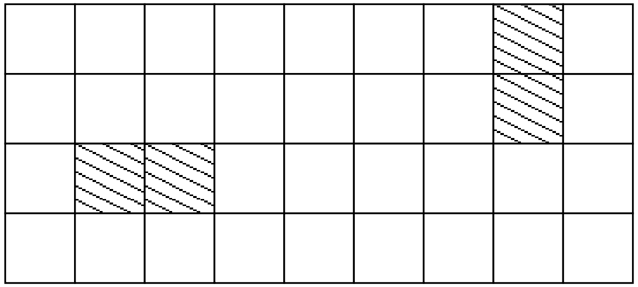
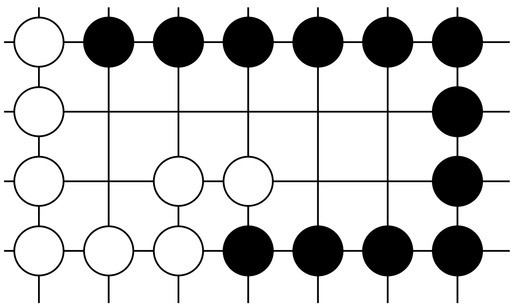
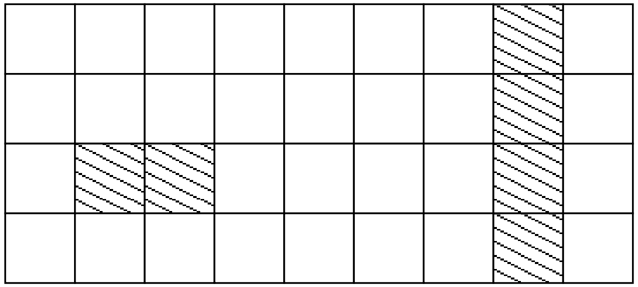




| Game | Mean | Temp |
|---|---|---|
| Go | 9/16 | 1 1/16 |
| Dom | 11/16 | 1 3/16 |

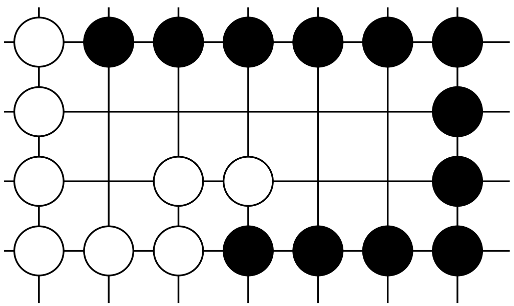
# Sample Game

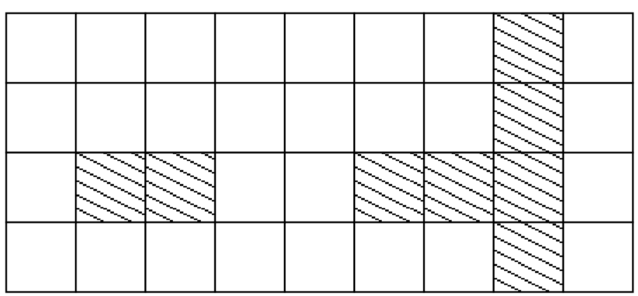




| Game | Mean | Temp |
|---|---|---|
| Go | 9/16 | 1 1/16 |
| Dom | 1 7/8 | 1 1/8 |

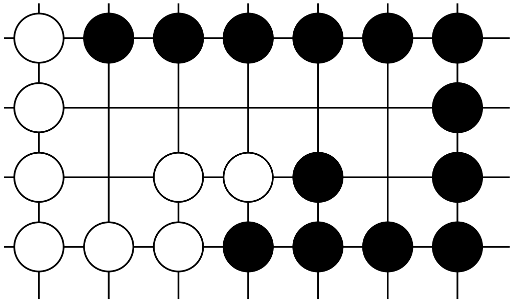
# Sample Game

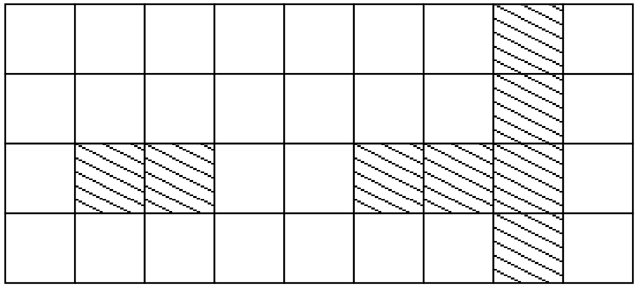




| Game | Mean | Temp |
|---|---|---|
| Go | 9/16 | 1 1/16 |
| Dom | 3/4 | 1 |

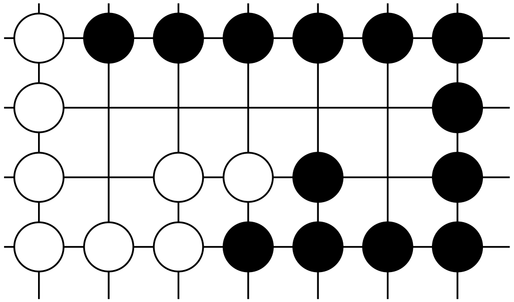
# Sample Game

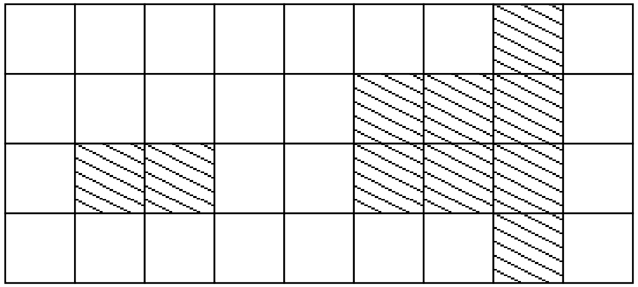




| Game | Mean | Temp |
|---|---|---|
| Go | 1 5/8 | 7/8 |
| Dom | 3/4 | 1 |

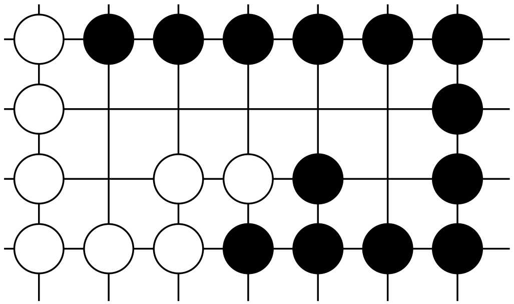
# Sample Game

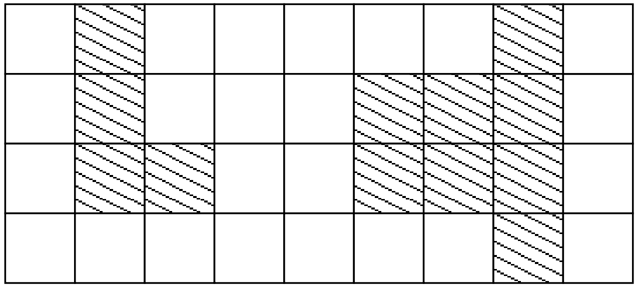




| Game | Mean | Temp |
|---|---|---|
| Go | 1 5/8 | 7/8 |
| Dom | -1/4 | 1 |

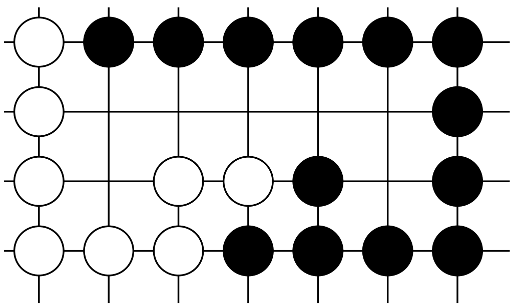
# Sample Game

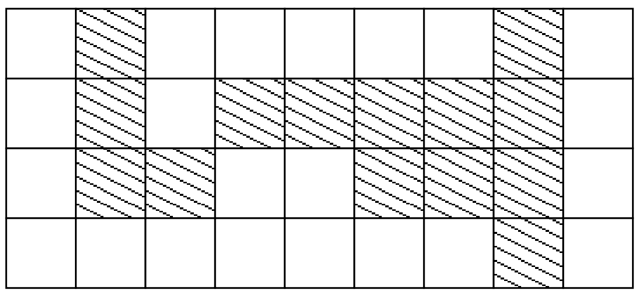




| Game | Mean | Temp |
|---|---|---|
| Go | 1 5/8 | 7/8 |
| Dom | 7/8 | 1 1/8 |

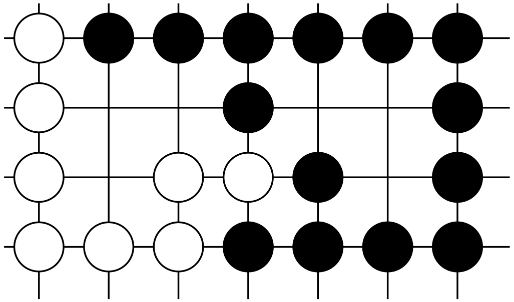
# Sample Game

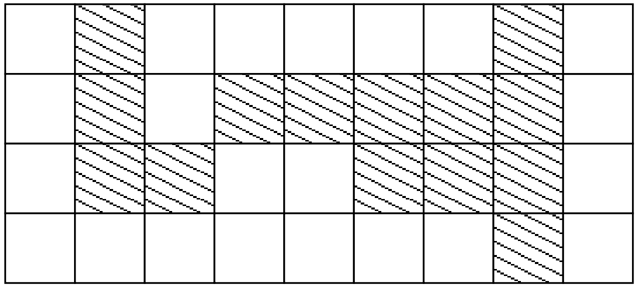




| Game | Mean | Temp |
|---|---|---|
| Go | 1 5/8 | 7/8 |
| Dom | -1/4 | 3/4 |

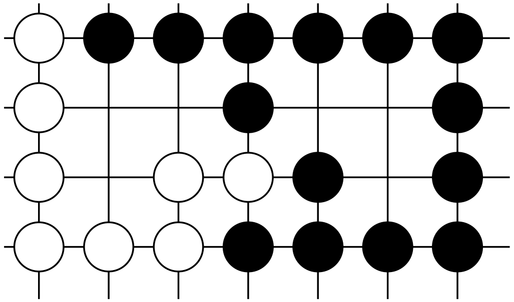
# Sample Game

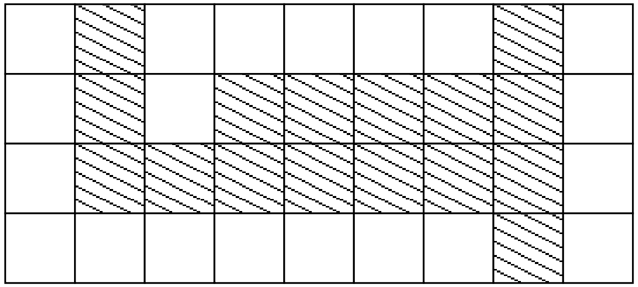




| Game | Mean | Temp |
|---|---|---|
| Go | 2 1/2 | 1/2 |
| Dom | -1/4 | 3/4 |

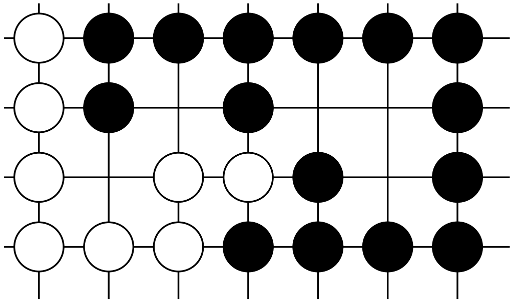
# Sample Game

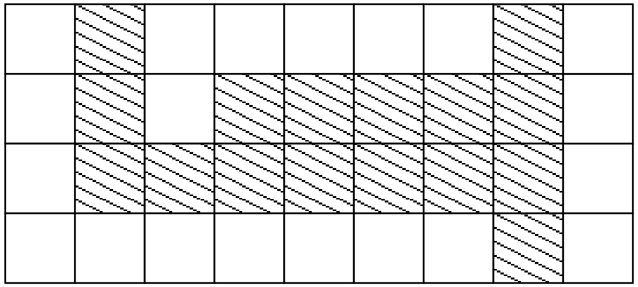




| Game | Mean | Temp |
|---|---|---|
| Go | 2 1/2 | 1/2 |
| Dom | -1 | -1/2 |

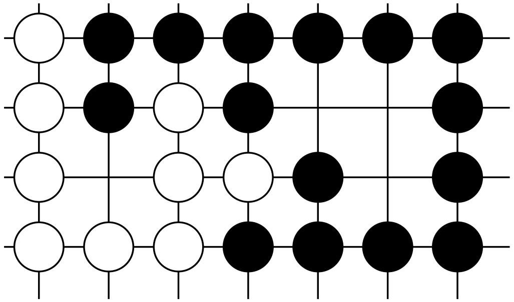
# Sample Game

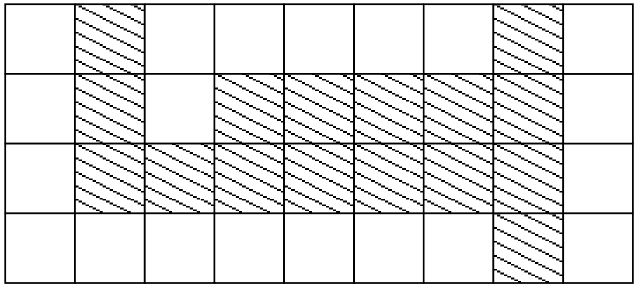




| Game | Mean | Temp |
| --- | --- | --- |
| Go | 3 | 0 |
| Dom | -1 | -1/2 |

# Sample Game





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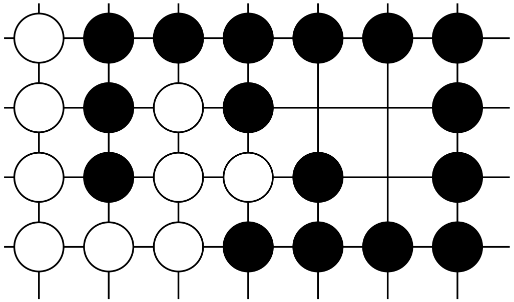

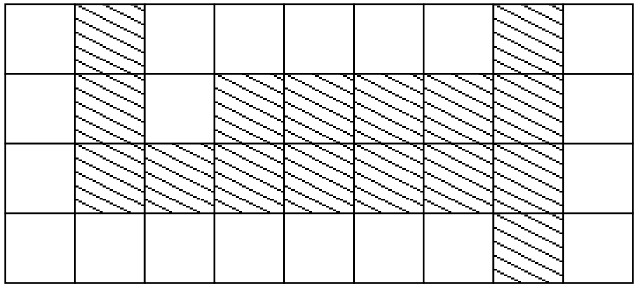

| Game | Mean | Temp |
|---|---|---|
| Go | 3 | -1 |
| Dom | -1 | -1/2 |

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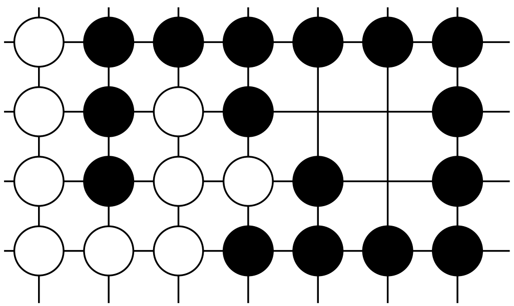

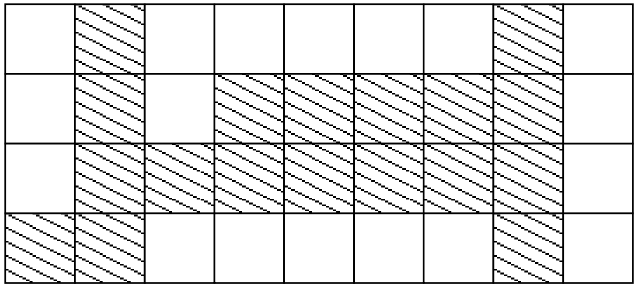

| Game | Mean | Temp |
|---|---|---|
| Go | 3 | -1 |
| Dom | -1/2 | -1/2 |

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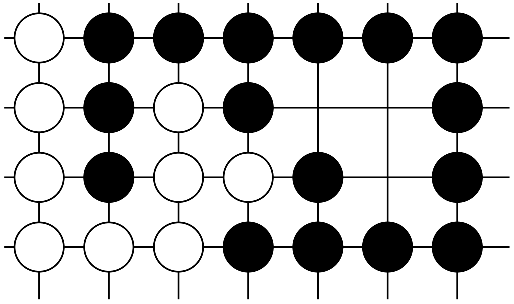
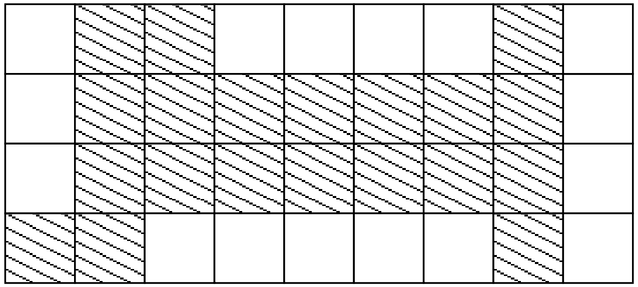


| Game | Mean | Temp |
|---|---|---|
| Go | 3 | -1 |
| Dom | -1 | -1 |



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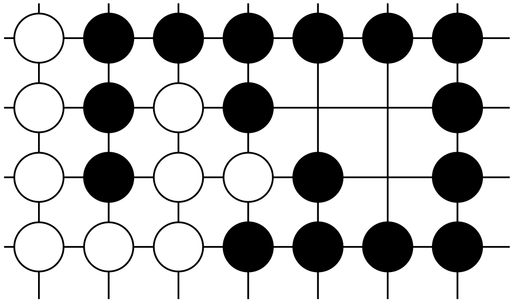

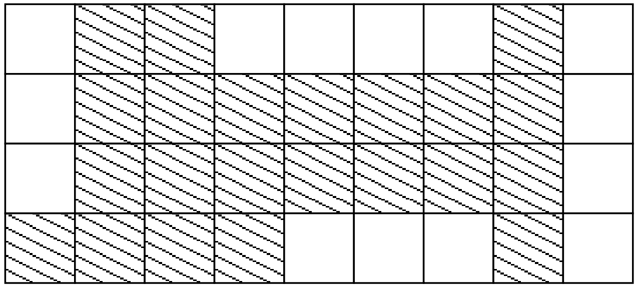

| Game | Mean | Temp |
|---|---|---|
| Go | 3 | -1 |
| Dom | 0 | -1 |

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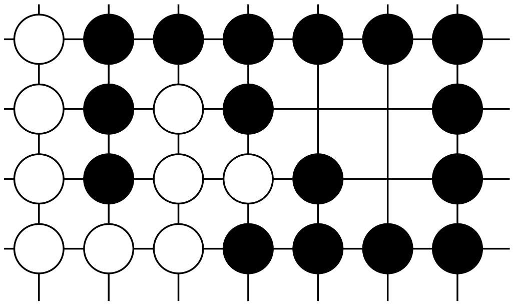
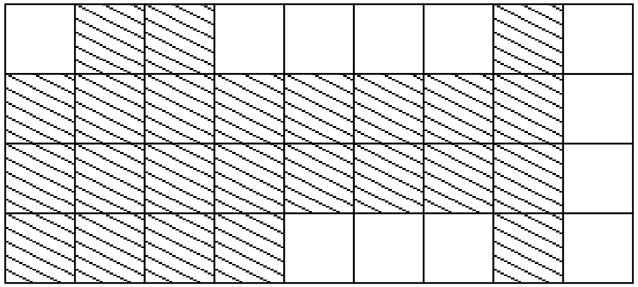




| Game | Mean | Temp |
|---|---|---|
| Go | 3 | -1 |
| Dom | -1 | -1 |

# Thermography

Classical:

- No loops (i.e. kos)
- Hill region at bottom, infinite mast at top

Goal: Extend to simple kos

# Economist Ko Rules

- Komaster is player who will win ko fight (if chooses to play it)
- Repeat positions not disallowed, but is critical ko if Komaster causes repeat
- Immediately after creating a critical ko, the Komaster is forced to play a local move

Results:

- Komaster is forced to win ko fight if plays it
- This is never disadvantageous

# Computing the Thermograph

Left and Right Scaffolds:

- LS $G_t = -t + max_{G^L}$(Right wall of $G_t^L$)
- RS $G_t = +t + min_{G^L}$(Left wall of $G_t^R$)

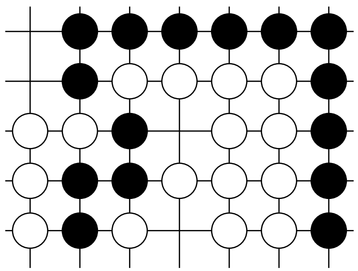
Left and Right Walls:

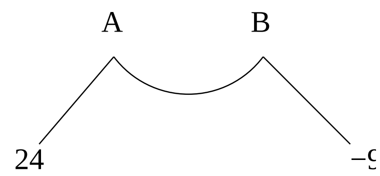
- If LS $\geq$ RS (hill region), same as scaffolds
- If LS $<$ RS (cave region), LW $=$ RW $=$ mast
- Intermediate masts may not be vertical

# Thermograph Properties

- Initial temperature
- Active/Dormant
- Activation temperature

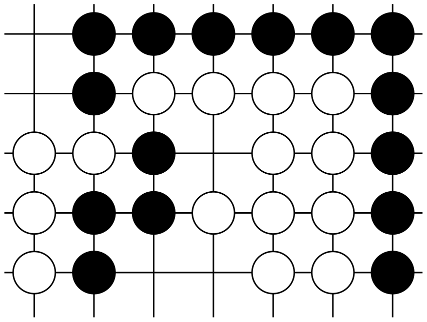
# Computing Ko Thermographs

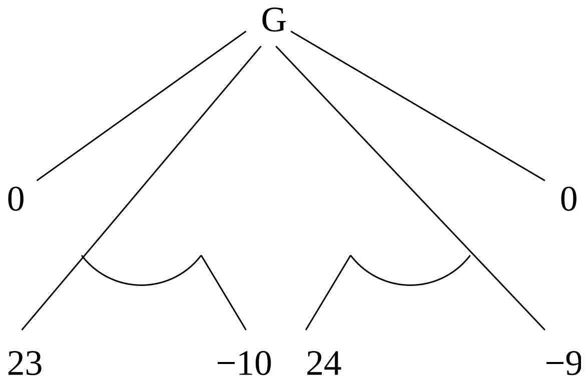


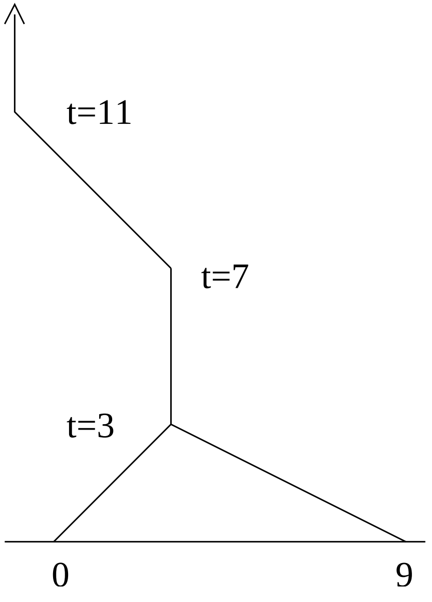



- $G = \{V | W, H\}$,
  $H = \{G, X | Y\}$
- $\tilde{G} = \{V | W\}$,
  $\hat{H} = \{\tilde{G}, X | Y\}$,
  $\hat{G} = \{V | W, \hat{H}\}$
- Creates a DAG based on who is Komaster

# Thousand-Year Ko





# More Thermograph Construction

- Given thermographs for $A$ and $B$, can construct thermograph for $\{A|B\}$
- Given thermographs for $A$ and $B$, cannot construct thermograph for $A + B$
- Means (infinite masts) add
- Many positions not relevant, top-down computation may be better

# Position Stability

- Stable: ancestors have higher temperatures
- Unstable: some ancestor has a lower temperature
- Semi-stable: some ancestor has same temperature, none have lower

Tradeoff:

- Handling semi-stable as stable is simpler and more efficient
- Handling it as unstable allows more precise calculation of infinitesimal values

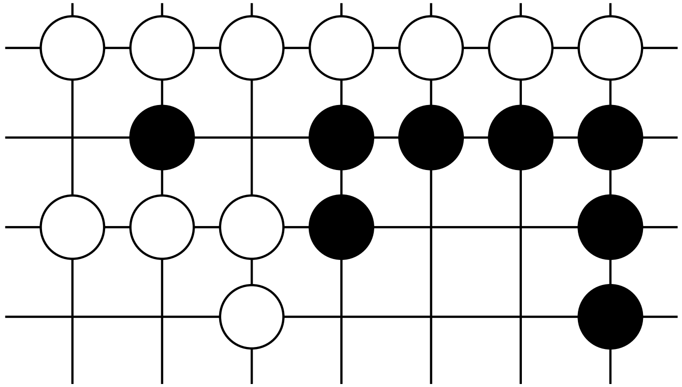
# Position Frequency

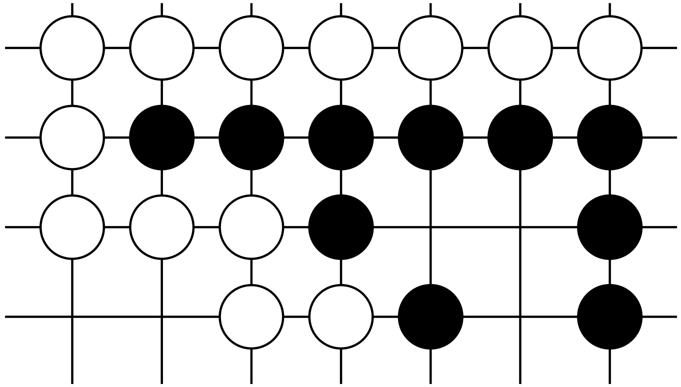
Given $n$ copies of a position, as $n \to \infty$:

- Mainline: position appears infinite number of times
- Sideline: position frequency is bounded

Mainline positions provide a good approximation of the game value in rich environments

# Mainline/Sideline Example





# Other Strategies

- Hotstrat - pick subgame with maximum temperature
- Thermostrat - pick subgame with widest hill at current temperature

Theorem: Thermostrat also optimal for Economist's rules

Observation: Thermostrat can exploit weaker opponents better than Sentestrat

# Economic Forecasting

- Estimate value using mean, adjusting for first player using half of current temperature
- Refine estimate with each temperature drop
- Beneficial to make the last move at the higher temperature
- Estimates reasonable, error bounded by $t/2$

# Conclusion

- Introduced concept of Komaster
- Extended thermographs to include basic kos
- Strategy to play optimally using thermographs (Economist rules)
- Ideas for more efficient construction of thermographs
- Refined estimates (forecasting)

# Any Questions?