Complexity Theory and Go

Go Seminar, University of Alberta

March 2003

Markus Enzenberger

~emarkus/web_docs/go-pspace/go-pspace-2up.ps

PSPACE

- A problem is in **PSPACE** if it can be solved with memory bounded by a polynomial of the problem size.
- A problem is **PSPACE-hard**, if every problem in PSPACE can be transformed in an equivalent instance of this problem.
- The transformation is called **reduction**.
  It must not take too much memory itself (e.g. logarithm of the problem size).
- PSPACE-hard problems are **computationally infeasible** for large problem sizes.

QBF

Quantified Boolean Formulas

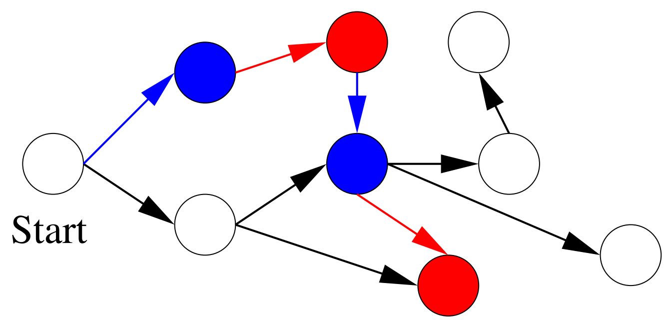
$\exists x_1 \forall x_2 \ldots \exists x_n : F(x_1, x_2, \ldots, x_n)$

- No free variables in F.
- F is in conjunctive normal form.

QBF: the problem whether a Quantified Boolean Formula is true.

QBF is known to be in PSPACE and PSPACE-hard.

GEOGRAPHY



- Two-player game played on a directed graph.
- Both players alternate in marking nodes.
- Each new marked node must be reachable from last node.
- The first player who cannot mark a node looses.

GEOGRAPHY: the problem whether the first player on a graph of size n has a winning strategy.

GEOGRAPHY is PSPACE-hard

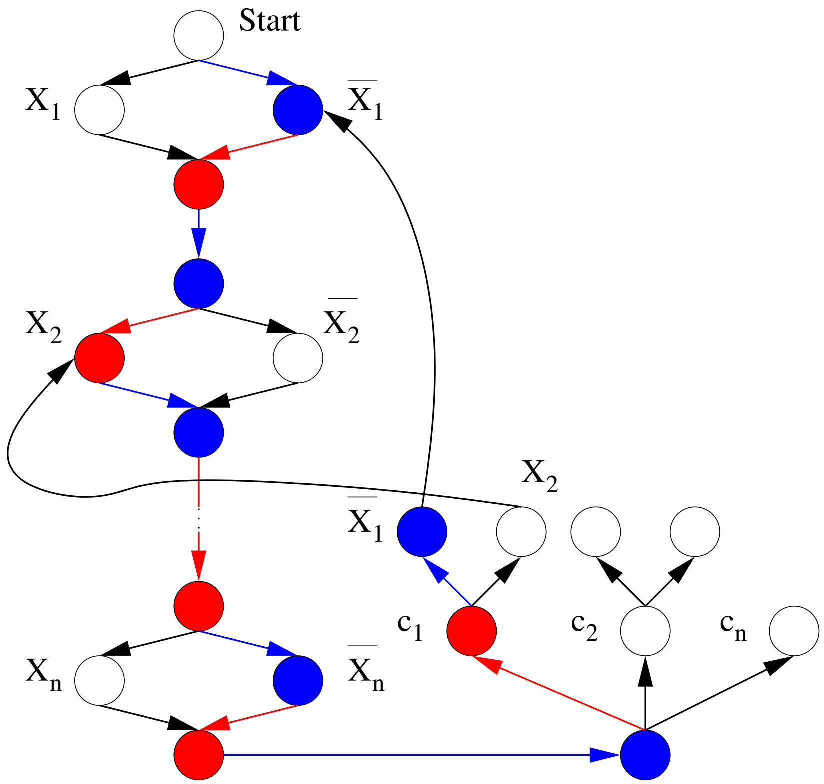
Proof idea:

Reduce QBF to GEOGRAPHY.

For an instance of QBF, construct an equivalent instance of GEOGRAPHY.

$\exists x_1 \forall x_2 \ldots \exists x_n : (\neg x_1 \vee x_2) \wedge \ldots$

GEOGRAPHY is PSPACE-hard, cont'd



- $\exists$ and $\forall$ assign values to $x_1 \ldots x_n$
- $\forall$ chooses a clause
- $\exists$ chooses a literal
- $\exists$ wins if and only if
  - its literal is assigned 1
  - every clause has a true literal
  - formula is true

GEOGRAPHY is PSPACE-hard, cont'd

- Reduction holds even when the formula does not simply alternate $\exists$ and $\forall$.
- It is possible to modify the graph, so that it is
  - **planar**: No crossing edges
  - **bipartite**: Every node can be reached only by one of the players
  - **of maximum degree 3**: Every node has 3 edges at maximum

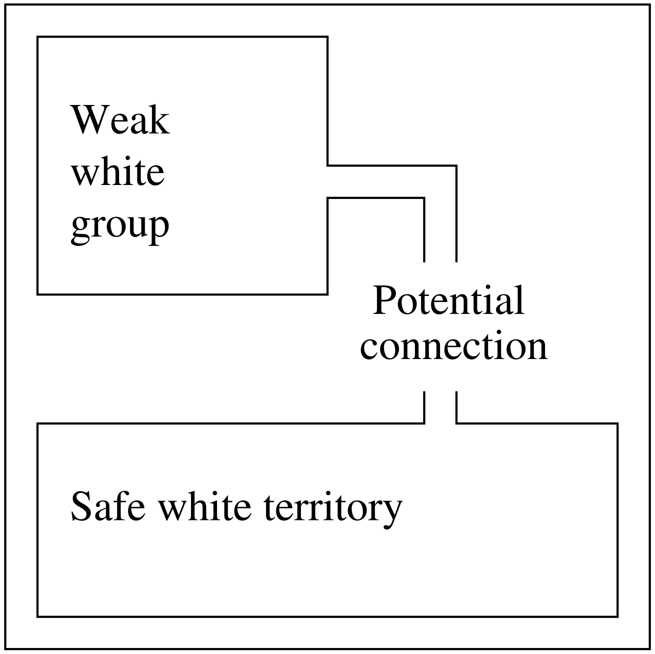
GO

The problem GO is defined to whether a player has a winning strategy in a given Go position of size $n \times n$.

We show that GO is PSPACE-hard by reducing GEOGRAPHY to GO.

Go position

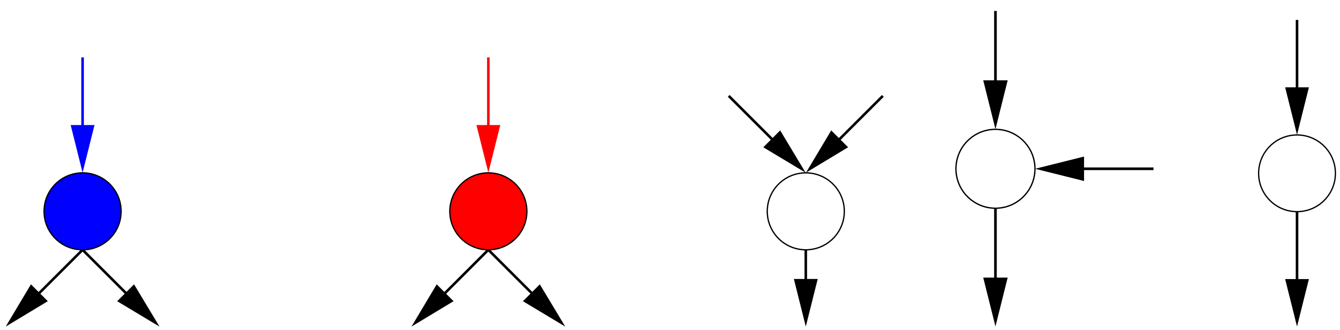
We construct a Go position in which White wins if and only if it can connect a large weak white group to a safe territory.



The potential connection will be constructed such that it is an encoding of the GEOGRAPHY graph.

Junction types

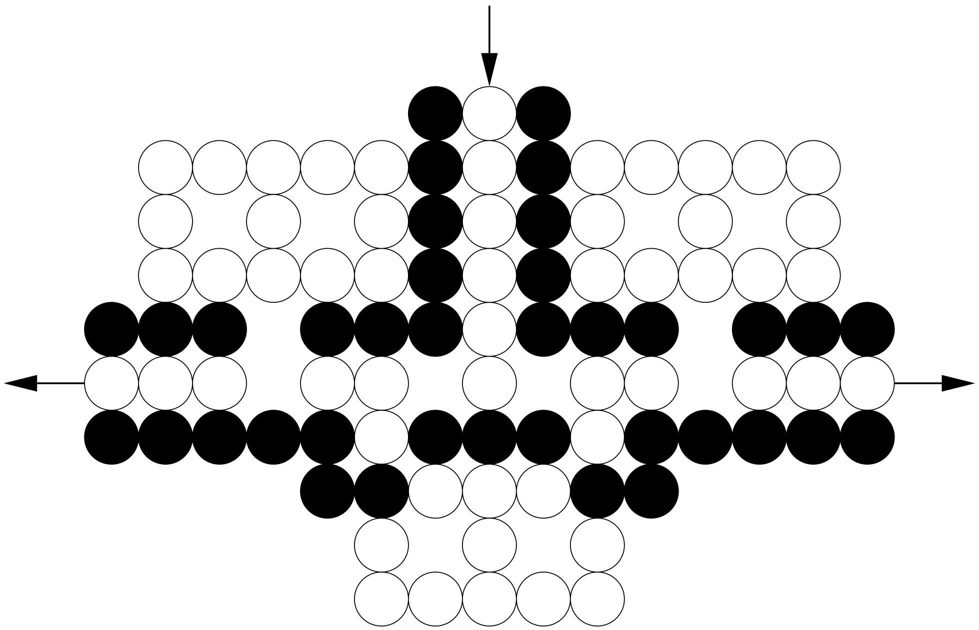
The following junctions occur in the graph:



$\exists$ player choice $\quad$ $\forall$ player choice $\quad$ Join $\quad$ Test $\quad$ Trivial

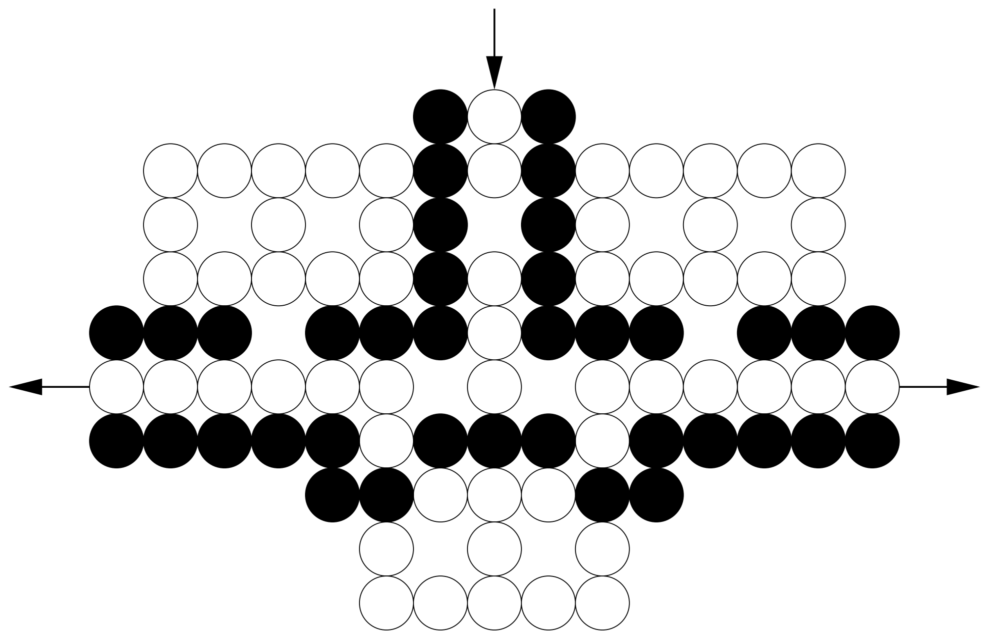
Encoding of the Trivial junction is not necessary
(we construct the junctions on the Go position such that it is always White to play at entering and leaving the junction).

∃ player choice



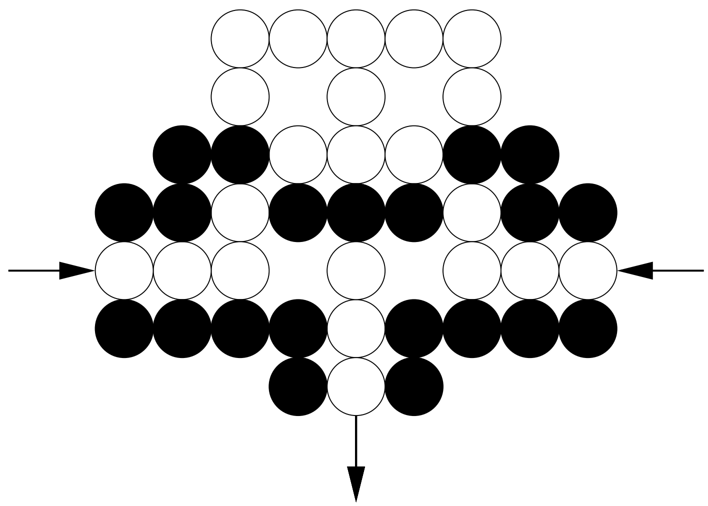
White can decide whether the play continues through the left- or right-hand pipe.
The other pipe will be closed.

$\forall$ player choice



Black can decide whether the play continues through the left- or right-hand pipe.
The other pipe will be closed.

Join

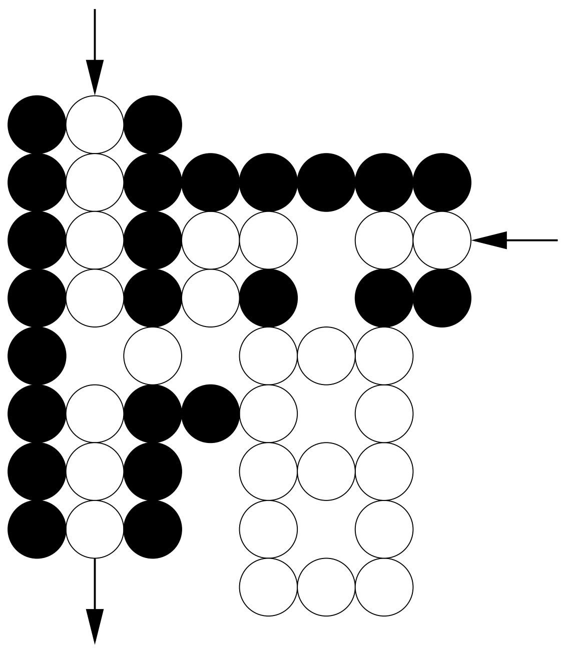


White enters through the left- or right-hand side.

The play continues downwards.

The other pipe will be closed.

Test



If White enters through the right-hand pipe then Black wins if and only if the play had previously passed down through the vertical pipe.

Remark

- If we modify the rules of Go such that there is a (polynomial) limit on the number of moves, then it is easy to show, that GO is in PSPACE.

References

Lichtenstein, D. and Sipser, M.:
Go is Polynominal-Space Hard.
Journal of the ACM, 27/2, 1980.

Papadimitriou, C.:
Complexity Theory
Addison Wesley, 1984.