2 Automating reasoning: Formal inference

Modelling mathematical reasoning

- Drawing *certain* conclusions from facts
- More facts → strictly more conclusions

(Note: not modelling plausible reasoning (yet):

- Drawing plausible conclusions from evidence
- More evidence → change conclusions)

First: Need a language to represent facts and conclusions

2.1 A simple first language: Language of propositions

- Primitive propositions $p, q, r, \ldots$
- Compound propositions
  - Logical symbols $\wedge, \vee, \neg, \rightarrow, \leftrightarrow, \bot, \top$
  - Composition: $\alpha \wedge \beta$, $\alpha \vee \beta$, $\neg\alpha$, $\alpha \rightarrow \beta$, $\alpha \leftrightarrow \beta$
    where α, β are propositions, either primitive or compound

2.2 Inference

Given a set of facts (propositions), what conclusions to draw? Let w = work, p = pass exam, f = fail course, u = understand concepts, a = do assignments.

Given	Infer ?	
$\{w \to p,\ w\}$	p	?
$\{e \to p \vee f,\ \neg f\}$	$e \to p$	?
$\{\mathit{over5ft} \to \mathit{over6ft},\ \mathit{over6ft}\}$	$\mathit{over5ft}$	?
$\{w \to p,\ p\}$	w	?
$\{w \to p,\ \neg p\}$	$\neg w$	?
$\{u \to (a \to p)\}$	$(u \to a) \to (u \to p)$	?
$\{w \to p\}$	$(p \to g) \to (w \to g)$	?
$\{p\}$	$\mathit{elvis\text{-}lives} \to p$	?

2.3 Formal inference

Conclusions drawn depend only on *logical form* of propositions

E.g., Formal rule of inference: *Modus Ponens*

Given $\{\alpha,\ \alpha \rightarrow \beta\}$, infer β

(written $\{\alpha,\ \alpha \rightarrow \beta\} \vdash \beta$ or $\dfrac{\alpha,\ \alpha \rightarrow \beta}{\beta}$)

Formal inference rules
– are automatable
– "pattern match" rules that depend only on logical form
– antecedent variables match existing propositions
– consequent variables produces new propositions

2.4 Two components of mechanized reasoning

Inference rules – encode domain independent rules of logical reasoning

Propositions – encode domain specific facts

2.5 Derivation

Starting with a set of propositions $A = \{\alpha_1, ..., \alpha_n\}$, can add new propositions β to A by applying available rules of inference. If a proposition γ can be added to A after a finite number of rule applications, then we say that γ is derivable from A; denoted $A \vdash \gamma$. If no finite number of rule applications can add γ to A, then γ is not derivable from A; denoted $A \not\vdash \gamma$.

Note that the derivability relation $\vdash$ depends on which inference rules are available.

2.6 E.g. application: automated question answering

Given domain facts $\{\alpha_1, ..., \alpha_n\} = A$, ask: is it the case that γ ?

If $A \vdash \gamma$ answer *yes*
If $A \vdash \neg\gamma$ answer *no*
If $A \not\vdash \gamma$ and $A \not\vdash \neg\gamma$ answer *I don't know*

E.g.
Given $\{$*lights_on* $\rightarrow$ *battery_ok, battery_ok* $\rightarrow$ *radio_works, lights_on*$\}$
is it the case that *radio_works* ?
is it the case that $\neg$*radio_works* ?

Given
$\{lights\_on \rightarrow battery\_ok,\ battery\_ok \wedge fuse\_ok \rightarrow radio\_works,\ lights\_on\}$
is it the case that $radio\_works$?
Given
$\{lights\_on \rightarrow battery\_ok,\ battery\_ok \wedge fuse\_ok \rightarrow radio\_works,\ lights\_on,\ fuse\_ok\}$
is it the case that $radio\_works$?
Given
$\{lights\_on \rightarrow battery\_ok,\ battery\_ok \wedge fuse\_ok \rightarrow radio\_works,\ lights\_on,\ \neg radio\_works\}$
is it the case that $\neg fuse\_ok$?
Given
$\{lights\_on \rightarrow battery\_ok,\ battery\_ok \wedge fuse\_ok \leftrightarrow radio\_works,\ lights\_on,\ radio\_works\}$
is it the case that $fuse\_ok$?

2.7 Is Modus Ponens adequate?

$$\{a,\ a \rightarrow b\} \quad \vdash \quad b$$

No! Cannot derive any of the following

$$\{a \rightarrow b, \neg b\} \quad \vdash \quad \neg a \text{ ?} \quad \textit{Modus Tollens } \frac{\alpha \rightarrow \beta,\ \neg\beta}{\neg\alpha}$$

$$\{a \wedge b \rightarrow c,\ a,\ b\} \quad \vdash \quad c \text{ ?} \quad \textit{And Introduction } \frac{\alpha,\ \beta}{\alpha \wedge \beta}$$

$$\{a \vee b \rightarrow c,\ a\} \quad \vdash \quad c \text{ ?} \quad \textit{Or Introduction } \frac{\alpha}{\alpha \vee \beta}$$

$$\{a \rightarrow b,\ \neg a \rightarrow c,\ b \rightarrow d,\ c \rightarrow d\} \quad \vdash \quad d \text{ ?} \quad \textit{Reasoning by cases } \frac{\alpha \rightarrow \beta,\ \neg\alpha \rightarrow \beta}{\beta}$$

$$\{\neg\neg a\} \quad \vdash \quad a \text{ ?} \quad \textit{Double Negation } \frac{\neg\neg\alpha}{\alpha}$$

2.8 Formal inference *system*

Set of inference rules
(plus, possibly, a restriction on the language)

***E.g.* 1: Modus Ponens**

***E.g.* 2: Resolution**

- Assumes propositions are in *clausal form*:

  $\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_k \vee q_1 \vee q_2 \vee \cdots \vee q_\ell$

  i.e., a disjunction of *literals*, where each *literal* is either p or $\neg p$

- Single rule of inference: Resolution rule

  $\dfrac{\alpha \vee \neg p, \quad \beta \vee p}{\alpha \vee \beta}$ (where α, β are also in clausal form)

  Note: special case when α, β are empty

  $\dfrac{\neg p, \quad p}{\bot}$ (contradiction)

- Generalizes Modus Ponens

  $\dfrac{\neg p \vee \beta, \ p}{\beta}$ $\left(\text{which is intuitively equivalent to } \dfrac{p \rightarrow \beta, \ p}{\beta}\right)$

  Note: we will often use intuitive equivalences

  $$\begin{aligned} \neg p \vee q \quad &\equiv \quad p \rightarrow q \\ \neg p_1 \vee \cdots \vee \neg p_k \vee q_1 \vee \cdots \vee q_\ell \quad &\equiv \quad p_1 \wedge \cdots \wedge p_k \rightarrow q_1 \vee \cdots \vee q_\ell \end{aligned}$$

  (You will be able to prove when and why these are equivalent later)

- *Strict clausal form*:

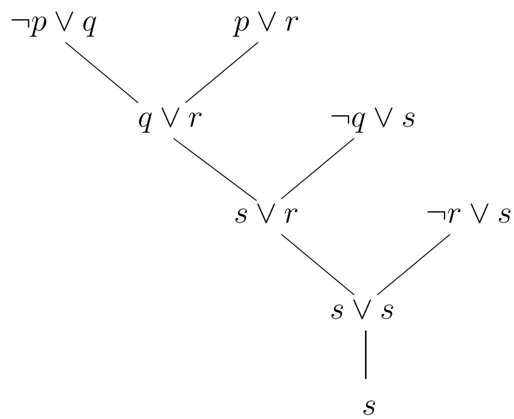
  - No repeated literals
  - No opposing literals
  - Simplification rules

    $\dfrac{\alpha \vee \neg p \vee \neg p}{\alpha \vee \neg p}$ $\dfrac{\alpha \vee q \vee q}{\alpha \vee q}$ $\dfrac{\alpha \vee \neg p \vee p}{\top}$ (just remove $\top$ clauses)

- Can reason by cases:

  E.g., Given $\{p \vee r,\ p \rightarrow q,\ q \rightarrow s,\ r \rightarrow s\}$, can derive s.

  Equivalent to $\{p \vee r,\ \neg p \vee q,\ \neg q \vee s,\ \neg r \vee s\}$,



- However, still missing some "reasonable" inferences?

  e.g., $\{\} \not\vdash \neg p \vee p$ under resolution

E.g. 3: Natural deduction system

Restrict propositions to any form using $\wedge, \vee, \rightarrow, \neg, \top, \bot$.

And elim $\dfrac{\alpha \wedge \beta}{\alpha,\ \beta}$ *And intro* $\dfrac{\alpha,\ \beta}{\alpha \wedge \beta}$

Or intro $\dfrac{\alpha}{\alpha \vee \beta}$

Impl elim $\dfrac{\alpha,\ \alpha \rightarrow \beta}{\beta}$ *Impl intro* $\dfrac{\text{If } A \cup \{\alpha\} \vdash \beta}{\alpha \rightarrow \beta}$

Reductio $\dfrac{\text{If } A \cup \{\alpha\} \vdash \bot}{\neg \alpha}$

Cases $\dfrac{\text{If } A \cup \{\alpha\} \vdash \beta \text{ and } A \cup \{\neg\alpha\} \vdash \beta}{\beta}$

Tautology $\dfrac{\alpha \vee \neg\alpha}{\top}$

Contradiction $\dfrac{\alpha,\ \neg\alpha}{\bot}$

E.g., given $\{p \to q,\ \neg p \to r,\ q \to s,\ r \to s\}$ can derive s.

1	$p \to q$		
2	$\neg p \to r$		
3	$q \to s$		
4	$r \to s$		
5.0		Assume p	
5.1		q	by Impl elim on 1 and 5.0
5.2		s	by Impl elim on 3 and 5.1
5	$p \to s$		by Impl intro
6.0		Assume $\neg p$	
6.1		r	by Impl elim on 2 and 6.0
6.2		s	by Impl elim on 4 and 6.1
6	$\neg p \to s$		by Impl intro
$7a.0$		Assume p	
$7a.1$		s	by Impl elim on 5 and $7a.0$
$7b.0$		Assume $\neg p$	
$7b.1$		s	by Impl elim on 6 and $7b.0$
7	s		by Cases

E.g., given $\{\}$ can derive $p \to p$

1.0		Assume p	
1.1		p	
1	$p \to p$		by Impl intro on 1.0 and 1.1

2.9 Characterizing inference systems

For a given inference system:

- Take a given set of propositions $A = \{\alpha_1, ..., \alpha_n\}$ and consider applying all available inference rules to A repeatedly:
- Get a monotonically growing set

  (Note: inference rules do not block each other, can always add conclusions in any order)

A set A is *closed* if no available inference rule can introduce any new propositions to A.

- The closure of a set A, $close(A)$, is called the *theory* of A.

- Monotonicity: $A \subset B$ implies that $close(A) \subset close(B)$
  (That is, adding new facts and new rules will only strictly increase the theory.)
- Monotonicity gives modularity: It is clear how new facts affect the theory. You never lose old conclusions. (This is a special feature of *logical* reasoning as opposed to *plausible* reasoning, which usually doesn't obey monotonicity.)

A proposition γ is called a *tautology* if $\{\} \vdash \gamma$. Such a γ is contained in every closure.

A set of propositions A is said to contain a *contradiction* if A contains any of $\perp$, $\top \rightarrow \perp$, or both α and $\neg\alpha$ for some α.

2.10 Computational complexity and search

Sometimes, even give that form of logical reasoning can be automated in principle, it can still be computationally hard to reach the desired conclusions. A surprising example of this is trying to prove the *"pigeonhole principle"* (that $N+1$ pigeons cannot be placed solitarily in N pigeonholes) using resolution:

E.g., 3 pigeons, 2 holes

		pigeons		
		A	B	C
holes	1	$A1$	$B1$	$C1$
	2	$A2$	$B2$	$C2$

Constraints:

$$\begin{array}{ccc} A1 \vee A2 & B1 \vee B2 & C1 \vee C2 \\ \neg(A1 \wedge B1) & \neg(A1 \wedge C1) & \neg(B1 \wedge C1) \\ \neg(A2 \wedge B2) & \neg(A2 \wedge C2) & \neg(B2 \wedge C2) \end{array}$$

Re-expressed in clausal form:

$$\begin{array}{ccc} A1 \vee A2 & B1 \vee B2 & C1 \vee C2 \\ \neg A1 \vee \neg B1 & \neg A1 \vee \neg C1 & \neg B1 \vee \neg C1 \\ \neg A2 \vee \neg B2 & \neg A2 \vee \neg C2 & \neg B2 \vee \neg C2 \end{array}$$

Exercise: derive $\perp$ from these facts using resolution.

Hint: it can be done, but it is surprisingly hard!

2.11 Readings

Burris, Chapter 1 and 2.
Dean, Allen, Aloimonos, Chapter 3.
Russell and Norvig 2nd Ed., Section 7.5.