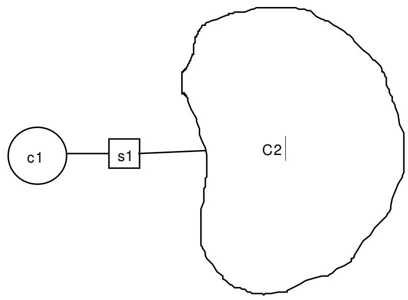
Normalization property for junction trees



Assume the given graphical model is a junction tree and that the message passing algorithm has been run (ensuring that the local consistency property holds between every pair of cliques and their separator). We show that the normalization constant for the entire junction tree must remain the same, even after we eliminate a leaf clique from the model.

Let $\mathcal{C}_1$ denote the complete set of cliques in the original graph, let Γ_1 denote the complete set of variables in the original graph, and let c_1 and s_1 denote a leaf clique and its separator respectively. We are going to eliminate the variables in $c_1 \setminus s_1$ from the normalization calculation. Let $\mathcal{C}_2 = \mathcal{C}_1 \setminus \{c_1\}$, and let Γ_2 denote the set of variables in $\mathcal{C}_2$. Then we have

$$\sum_{\mathbf{x}_{\Gamma_1}} \Psi_{\mathcal{C}_1}(\mathbf{x}_{\Gamma_1}) = \sum_{\mathbf{x}_{c_1 \setminus s_1}} \sum_{\mathbf{x}_{s_1}} \sum_{\mathbf{x}_{\Gamma_1 \setminus c_1}} \frac{\psi_{c_1}(\mathbf{x}_{c_1})\ \Psi_{\mathcal{C}_2}(\mathbf{x}_{\Gamma_2})}{\phi_{s_1}(\mathbf{x}_{s_1})}$$

$$= \sum_{\mathbf{x}_{s_1}} \frac{1}{\phi_{s_1}(\mathbf{x}_{s_1})} \sum_{\mathbf{x}_{c_1 \setminus s_1}} \sum_{\mathbf{x}_{\Gamma_1 \setminus c_1}} \psi_{c_1}(\mathbf{x}_{c_1})\ \Psi_{\mathcal{C}_2}(\mathbf{x}_{\Gamma_2})$$

Note that by the junction tree property $\Gamma_1 \setminus c_1 = \Gamma_2 \setminus s_1$ and furthermore $(c_1 \setminus s_1) \cap \Gamma_2 = \emptyset$, so we obtain

$$= \sum_{\mathbf{x}_{s_1}} \frac{1}{\phi_{s_1}(\mathbf{x}_{s_1})} \left[\sum_{\mathbf{x}_{c_1 \setminus s_1}} \psi_{c_1}(\mathbf{x}_{c_1}) \right] \left[\sum_{\mathbf{x}_{\Gamma_2 \setminus s_1}} \Psi_{\mathcal{C}_2}(\mathbf{x}_{\Gamma_2}) \right]$$

We assumed local consistency between ψ_{c_1} and ϕ_{s_1}, hence

$$= \sum_{\mathbf{x}_{s_1}} \sum_{\mathbf{x}_{\Gamma_2 \setminus s_1}} \Psi_{\mathcal{C}_2}(\mathbf{x}_{\Gamma_2})$$

$$= \sum_{\mathbf{x}_{\Gamma_2}} \Psi_{\mathcal{C}_2}(\mathbf{x}_{\Gamma_2})$$