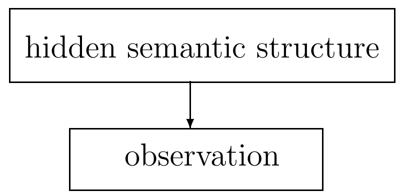
# 15 Interpreting senses (perception)

We use probabilistic models to build interpretation systems.

**Generic model**




**Interpretation**

$$\text{interp} \;=\; \arg \max_{\text{possible truth}} \text{P}(\text{possible truth}|\text{observation})$$

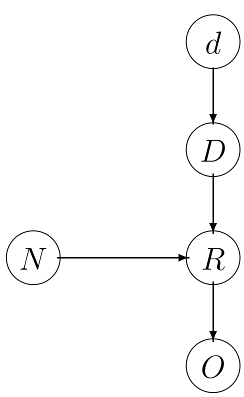
## 15.1 Toy example: Handwritten digit recognition

(This is Assignment 3!)

Given an image, recognize the digit

We consider two generative probabilistic models

**Generative Model 1**




Pick a digit $d \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$
(with uniform probability for each digit)

Get corresponding image template (a matrix)

$$D = \begin{bmatrix} d_{11} & \dots & d_{18} \\ \vdots & \ddots & \vdots \\ d_{81} & \dots & d_{88} \end{bmatrix} \in \{1, 2, 3, 4\}^{8\times 8}$$

Generate a noise matrix

$$N = \begin{bmatrix} n_{11} & \dots & n_{18} \\ \vdots & \ddots & \vdots \\ n_{81} & \dots & n_{88} \end{bmatrix} \in \{-3, -2, -1, 0, 1, 2, 3\}^{8\times 8}$$

Add noise matrix to template to generate raw image $r_{ij} = d_{ij} + n_{ij}$

$$R = \begin{bmatrix} r_{11} & \dots & r_{18} \\ \vdots & \ddots & \vdots \\ r_{81} & \dots & r_{88} \end{bmatrix} \in \{-2,-1,0,1,2,3,4,5,6,7\}^{8\times 8}$$

Threshold raw image values to obtain final observed image

$$o_{ij} = \begin{cases} r_{ij} & \text{if } 1 \le r_{ij} \le 4 \\ 1 & \text{if } r_{ij} < 1 \\ 4 & \text{if } r_{ij} > 4 \end{cases}$$

$$O = \begin{bmatrix} o_{11} & \dots & o_{18} \\ \vdots & \ddots & \vdots \\ o_{81} & \dots & o_{88} \end{bmatrix} \in \{1,2,3,4\}^{8\times 8}$$

**Note** Once the digit $d$ has been selected, each pixel is generated independently. That is, the pixels are conditionally independent *given* the digit identity $d$.

### Recognition with Model 1

Given an image $O$, compute

$$\begin{aligned} d^* &= \arg\max_d \mathrm{P}(d|O) \\ &= \arg\max_d \frac{\mathrm{P}(dO)}{\mathrm{P}(O)} \\ &= \arg\max_d \mathrm{P}(dO) \\ &\quad \text{since } \mathrm{P}(O) \text{ is a constant with respect to } d \end{aligned}$$

Now note that $\mathrm{P}(dO)$ is a marginal

$$\begin{aligned}
\mathrm{P}(dO) &= \sum_D \sum_N \sum_R \mathrm{P}(dDNRO) \\
&= \sum_D \sum_N \sum_R \mathrm{P}(d)\ \mathrm{P}(DNRO|d) \\
&= \sum_D \sum_N \sum_R \mathrm{P}(d)\ \prod_{i,j} \mathrm{P}(d_{ij}, n_{ij}, r_{ij}, o_{ij}|d) \\
&\quad \text{since pixels are conditionally independent given } d \\
&= \sum_D \sum_N \sum_R \mathrm{P}(d) \prod_{i,j} \mathrm{P}(d_{ij}|d)\ \mathrm{P}(n_{ij})\ \mathrm{P}(r_{ij}|d_{ij}, n_{ij})\ \mathrm{P}(o_{ij}|r_{ij}) \\
&\quad \text{by the Bayesian network structure behind each pixel}
\end{aligned}$$

But notice

$$\begin{aligned}
\mathrm{P}(d_{ij}|d) &= \begin{cases} 1 & \text{if } d_{ij} = d_{ij}^{(d)} \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{P}(r_{ij}|d_{ij}, n_{ij}) &= \begin{cases} 1 & \text{if } r_{ij} = d_{ij} + n_{ij} \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{P}(o_{ij}|r_{ij}) &= \begin{cases} 1 & \text{if } r_{ij} = o_{ij} \\ & \quad \text{or } r_{ij} < 1 \text{ and } o_{ij} = 1 \\ & \quad \text{or } r_{ij} > 4 \text{ and } o_{ij} = 4 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

So combining the last two of these yields

$$\begin{aligned}
\mathrm{P}(o_{ij}|d_{ij}, n_{ij}) &= \sum_{r_{ij}} \mathrm{P}(o_{ij}, r_{ij}|d_{ij}, n_{ij}) \\
&= \sum_{r_{ij}} \mathrm{P}(o_{ij}|r_{ij}, d_{ij}, n_{ij})\mathrm{P}(r_{ij}|d_{ij}, n_{ij}) \\
&= \sum_{r_{ij}} \mathrm{P}(o_{ij}|r_{ij})\mathrm{P}(r_{ij}|d_{ij}, n_{ij}) \\
&\quad \text{by the Bayesian network structure} \\
&= \begin{cases} 1 & \text{if } d_{ij} + n_{ij} = o_{ij} \\ & \quad \text{or } d_{ij} + n_{ij} < 1 \text{ and } o_{ij} = 1 \\ & \quad \text{or } d_{ij} + n_{ij} > 4 \text{ and } o_{ij} = 4 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}$$

Thus we have

$$\begin{aligned}
\mathrm{P}(dO) &= \sum_N \mathrm{P}(d) \prod_{i,j} \mathrm{P}(n_{ij})\ \mathrm{P}(o_{ij}|d_{ij}^{(d)}, n_{ij}) \\
&= \mathrm{P}(d) \sum_{n_{11}} \cdots \sum_{n_{88}} \prod_{i,j} \mathrm{P}(n_{ij})\ \mathrm{P}(o_{ij}|d_{ij}^{(d)}, n_{ij}) \\
&= \mathrm{P}(d) \prod_{i,j} \sum_{n_{ij}} \mathrm{P}(n_{ij})\ \mathrm{P}(o_{ij}|d_{ij}^{(d)}, n_{ij}) \\
&\quad \text{distributing the sums over the products}
\end{aligned}$$

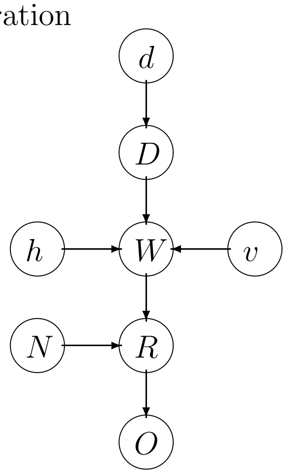
Now, to calculate conditional probabilities, note that for each $d = 0, 1, 2, ..., 9$, we can just use

$$P(d|O) = \frac{P(dO)}{P(O)} = \frac{P(dO)}{\sum_{d'=0}^{9} P(d'O)}$$

**Generative Model 2**

Add a global shear transformation over the whole image.

Generation




template

warped image $W = \text{shear}(D, h, v)$

raw image $R = W + N$

thresholded image $O = \text{threshold}(R)$

Here the shear variables $h$ and $v$ are independently chosen from $h \in \{0, 1, 2\}$ and $v \in \{-1, 0, 1\}$ respectively, and the shear operation is defined by

$$w_{ij} = \begin{cases} d_{\,t(j,i,v)\; t(i,j,h)} & \text{if both } 1 \le t(j,i,v) \le 8 \text{ and } 1 \le t(i,j,h) \le 8 \\ 1 & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} t(j,i,v) &= i + round(v(5-j)/4) \\ t(i,j,h) &= j + round(h(5-i)/4) \end{aligned}$$

**Recognition with Model 2**

First note that pixels are no longer conditionally independent given $d$. They are conditionally independent given $d$, $h$ and $v$. The pixels $w_{ij}$, $r_{ij}$ and $o_{ij}$ are deterministic functions

$$\begin{aligned} w_{ij} &= d_{\,t(j,i,v)\; t(i,j,h)} \quad \text{where } t(j,i,v) \text{ and } t(i,j,h) \text{ are as above} \\ r_{ij} &= n_{ij} + r_{ij} \\ o_{ij} &= \text{threshold}(n_{ij} + w_{ij}) \end{aligned}$$

Therefore, as with Model 1, we can simplify the Bayesian network by "collapsing" the deterministic functions into $O$ and marginalizing out $W$ and $R$, yielding

$$\begin{aligned}
\mathrm{P}(dO) &= \sum_h \sum_v \sum_D \sum_N \mathrm{P}(dhvDNO) \\
&= \sum_h \sum_v \sum_D \sum_N \mathrm{P}(dhv)\ \mathrm{P}(DNO|dhv) \\
&= \sum_h \sum_v \sum_D \sum_N \mathrm{P}(dhv) \prod_{i,j} \mathrm{P}(d_{ij}, n_{ij}, o_{ij}|d, h, v) \\
&\quad \text{since pixels are conditionally independent given } d,\ h \text{ and } v \\
&= \sum_h \sum_v \sum_D \sum_N \mathrm{P}(d)\ \mathrm{P}(h)\ \mathrm{P}(v) \prod_{i,j} \mathrm{P}(d_{ij}|d)\ \mathrm{P}(n_{ij})\ \mathrm{P}(o_{ij}|d_{ij}, h, v, n_{ij}) \\
&\quad \text{by the Bayesian network structure} \\
&= \sum_h \sum_v \sum_N \mathrm{P}(d)\ \mathrm{P}(h)\ \mathrm{P}(v) \prod_{i,j} \mathrm{P}(n_{ij})\ \mathrm{P}(o_{ij}|d_{ij}^{(d)}, h, v, n_{ij}) \\
&\quad \text{where } \mathrm{P}(o_{ij}|d_{ij}^{(d)}, h, v, n_{ij}) = \begin{cases} 1 & \text{if } o_{ij} = \text{threshold}\left(n_{ij} + d^{(d)}_{t(j,i,v)\ t(i,j,h)}\right) \\ 0 & \text{otherwise} \end{cases} \\
&= \mathrm{P}(d) \sum_h \mathrm{P}(h) \sum_v \mathrm{P}(v) \sum_N \prod_{i,j} \mathrm{P}(n_{ij})\ \mathrm{P}(o_{ij}|d_{ij}^{(d)}, h, v, n_{ij}) \\
&= \mathrm{P}(d) \sum_h \mathrm{P}(h) \sum_v \mathrm{P}(v) \prod_{i,j} \sum_{n_{ij}} \mathrm{P}(n_{ij})\ \mathrm{P}(o_{ij}|d_{ij}^{(d)}, h, v, n_{ij}) \\
&\quad \text{by distributing sums over products}
\end{aligned}$$

Conditional probabilities $P(d|O)$ can be calculated in the same way as for Model 1.

## Readings

Russell and Norvig 2nd Ed: Chapter 24
Dean, Allen, Aloimonos: Chapter 9