9 Correct/exhaustive first order inference

Given a first order formal inference system

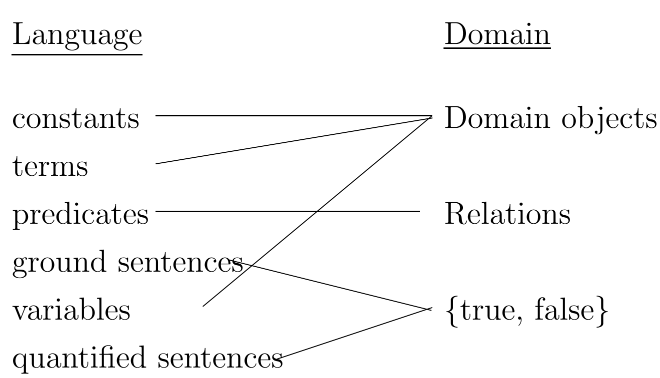
- Are the formal inference rules correct?
- Is the formal inference system exhaustive?

Same strategy as propositional logic:

- Create an independent evaluation scheme
  - Specify possible states of affairs
  - Assign truth values to atomic sentences
  - Recursively evaluate compound sentences
- Then try to show
  - **Correctness** $A \vdash \gamma$ implies $A \models \gamma$
  - **Exhaustiveness** $A \models \gamma$ implies $A \vdash \gamma$

9.1 Possible state of affairs: A structure

Map language elements to a possible domain and relations



9.2 Structures

We evaluate sentences by referring to a given structure $I = (D, C, F, R)$

D a set
C a function: constant symbols $\to D$
F a function: functions symbols $\to (D^n \to D)$
R a function: predicate symbols $\to S \subseteq D^n$

Given such a structure I, we can begin to evaluate sentences as follows

Ground terms can be evaluated recursively to a specific element of D

E.g., for constant symbols a, b and function symbol f, we obtain
- $I(a) = C(a) =$ a specific object in D, and
- $I(f(a, b)) = F(f)(C(a), C(b)) =$ a specific object in D

Predicate symbols are assigned a specific relation $S \subset D^n$

E.g., for predicate symbol P we obtain
- $I(P) = R(P) =$ a specific set of tuples $\{\langle d_{11}...d_{1n}\rangle, \langle d_{21}...d_{2n}\rangle, ...\}$

9.3 Evaluating ground sentences

Atomic ground sentences are assigned true or false, depending on whether the tuple of arguments is in the predicate symbol's assigned relation

$$I(P(t_1, ..., t_k)) = \text{true} \quad \text{iff} \quad \langle I(t_1), ..., I(t_k)\rangle \in I(P)$$

Compound ground sentences are evaluated recursively using the same rules as propositional logic

$$\begin{aligned} I(\neg\alpha) = \text{true} &\quad \text{iff} \quad I(\alpha) = \text{false} \\ I(\alpha \wedge \beta) = \text{true} &\quad \text{iff} \quad I(\alpha) = \text{true and } I(\beta) = \text{true} \\ I(\alpha \vee \beta) = \text{true} &\quad \text{iff} \quad I(\alpha) = \text{true or } I(\beta) = \text{true} \\ I(\alpha \to \beta) = \text{false} &\quad \text{iff} \quad I(\alpha) = \text{true and } I(\beta) = \text{false} \end{aligned}$$

9.4 Evaluating *quantified* sentences

We first need to introduce an auxiliary structure in addition to I

Variable assignment V: variables $\rightarrow D$

Given a structure I and a variable assignment V we can now evaluate open formulas because the assigned variables can now be treated like constants. First to evaluate atomic formulas we use

$$I_V(P(t_1, \ldots, t_n)) = \text{true} \quad \text{iff} \quad \langle I_V(t_1), \ldots, I_V(t_n) \rangle \in R(P)$$

Next to evaluate compound formulas (without quantifiers) we use the same recursive rules as above

$$\text{E.g.,} \quad I_V(\neg\alpha) = \text{true} \;\; \text{iff} \;\; I_V(\alpha) = \text{true}, \quad \text{etc.}$$

Then using these auxiliary variable assignments, we can evaluate quantified sentences as follows.

Universally quantified sentences

$I_V(\forall x\ \varphi(...x...)) = \text{true}$ iff $I_U(\varphi(...x...)) = \text{true}$ for all variable assignments U that agree with V except possibly on x

$I(\forall x\ \varphi) = \text{true}$ iff $I_V(\forall x\ \varphi) = \text{true}$ for all variable assignments V

Existentially quantified sentences

$I_V(\exists x\ \varphi(...x...)) = \text{true}$ iff $I_U(\varphi(...x...)) = \text{true}$ for some variable assignment U that agrees with V except possibly on x

$I(\exists x\ \varphi) = \text{true}$ iff $I_V(\exists x\ \varphi) = \text{true}$ for some variable assignment V

Therefore, given an interpretation I, we can evaluate any sentence.

9.5 Terminology

Same as propositional logic

- I *satisfies* α if $I(\alpha) = \text{true}$
- I *falsifies* α if $I(\alpha) = \text{false}$
- α is *satisfiable* if exists I such that $I(\alpha) = \text{true}$
- α is *falsifiable* if exists I such that $I(\alpha) = \text{false}$
- α is *unsatisfiable* (or *inconsistent*) if $I(\alpha) = \text{false}$ for all I
- α is *unfalsifiable* (or *valid*) if $I(\alpha) = \text{true}$ for all I
- α *entails* β if every I that makes α evaluate to true, makes β evaluate to true as well. Written $\alpha \models \beta$.

9.6 Resolution is correct

Recall the resolution rule for first order logic

$$\frac{\alpha \to p(\underline{v}) \vee \beta \quad \gamma \wedge p(\underline{v}) \to \delta}{\alpha \wedge \gamma \to \beta \vee \delta}$$

As with propositional logic, we must show that any structure I that makes the antecedents $\alpha \to p(\underline{v}) \vee \beta$ and $\gamma \wedge p(\underline{v}) \to \delta$ evaluate to true, must also make the consequent $\alpha \wedge \gamma \to \beta \vee \delta$ evaluate to true.

Proof. Same proof as with propositional logic. That is, assume a structure I that makes both antecedents evaluate to true, and consider the two cases $I(p(\underline{v})) = \text{true}$ and $I(p(\underline{v})) = \text{false}$. Argue that in each case I must force $\alpha \wedge \gamma \to \beta \vee \delta$ to evaluate to true. ■

9.7 Specialization is correct

Recall the specialization rule for first order logic

$$\frac{\alpha}{[\alpha]_{x/t}}$$

We must show that any structure I that makes the antecedent α evaluate to true, must also make the consequent $[\alpha]_{x/t}$ evaluate to true.

Proof. Assume $I(\alpha) = \text{true}$. Hence, $I_V(\alpha) = \text{true}$ for all V. We want to show that $I_U([\alpha]_{x/t}) = \text{true}$ for all U.

Let U be an arbitrary assignment, and let $d = I_U(t)$. Let V' be an assignment that agrees with U on all variables except x, and $I_{V'}(x) = d$. Then $I_{V'}(\alpha) = I_U([\alpha]_{x/t})$. By assumption, $I_{V'}(\alpha) = \text{true}$, so $I_U([\alpha]_{x/t}) = \text{true}$. ■

9.8 Exhaustiveness w.r.t. deriving contradictions

The formal inference system resolution+specialization+simplification is exhaustive w.r.t. deriving contradictions. That is, if A is unsatisfiable, then $A \vdash \top \rightarrow \bot$.

Proof. (sketch) Let A be a set of sentences in conjunctive normal form. First we need two definitions

Herbrand universe of A	=	ground terms constructable from the constant and function symbols mentioned in A
Herbrand base of A	=	set of all ground sentences constructible by using ground terms in Herbrand universe of A

Lemma (Herbrand's theorem) If A is unsatisfiable, then $HB(A)$ is unsatisfiable.

Lemma (Compactness theorem) If B is unsatisfiable, then there exists a *finite* subset $B' \subseteq B$ such that B' is unsatisfiable.

Now to prove the theorem, assume A is unsatisfiable. Then by Herbrand's theorem $HB(A)$ must be unsatisfiable, and by the compactness theorem there must be some finite subset H of $HB(A)$ that is unsatisfiable. Now realize that $A \vdash H$ just by applying substitution steps. Finally, if H is unsatisfiable then $H \vdash \top \rightarrow \bot$ by using resolution steps (using the same argument as for propositional resolution). ■

Readings

Russell and Norvig 2nd Ed.: Chapter 9
Genesereth and Nilsson: Sections 2.3, 4.10
Burris: Chapter 4 (especially 4.10)