Strings

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Programming Club Meeting
Outline

- Suffix Arrays
- Knuth-Morris-Pratt Pattern Matching
Suffix Arrays (no code, see Comp. Prog. text)

Sort all of the *suffixes* of a string lexicographically.

banana
- aban
- an
- anaban
- ananaban
- ban
- bananaban
- n
- naban
- nanaban
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Suffix Array

An array of indices of the start positions of the suffixes in sorted order.

Example

For string banana

```
int sarray[] = {5, 7, 3, 1, 6, 0, 8, 4, 2}
```
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This is often overkill in the contest setting and a bit technical, let’s see an $O(n \log^2 n)$ algorithm.

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**Suffix Array**
An array of indices of the start positions of the suffixes in sorted order.

**Example**
For string `bananaban`
int sarray[] = {5, 7, 3, 1, 6, 0, 8, 4, 2};
Idea: For $i = 0, \ldots, \log_2 n$, sort the suffixes just by their first $2^i$ characters.

$i = 0$

- ananaban
- anaban
- aban
- an
- bananaban
- ban
- nanaban
- naban
- n

Can do in $O(n \log n)$ time (recall we are actually just sorting the indices, not the whole suffixes).
Next, sort the suffixes by their length 2 prefixes.

- aban
- ananaban
- anaban
- an

- banananaban
- ban
- n
- nanaban
- naban
Next, sort the suffixes by their length 4 prefixes.

- aban
- an
- anaban
- ananaban
- ban
- bananaban
- n
- naban
- nanaban

To check if nanaban < naban, just look up the 2nd half of the red parts to see how they were ordered last step.
Generally, to sort the suffixes by their length $2^{i+1}$ prefixes we check $<$ using the ordering based on length $2^i$ prefixes.
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Check how the first $2^i$ characters of two suffixes $a, b$ compare using the previous ordering. If they are different then just return that result.

If they are the same, check how the second $2^i$ characters of $a, b$ compare again using the previous ordering.
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Check how the first $2^i$ characters of two suffixes $a$, $b$ compare using the previous ordering. If they are different then just return that result.

If they are the same, check how the second $2^i$ characters of $a$, $b$ compare again using the previous ordering.

**Example**

`nanaban` vs. `naban`. Length-2 prefixes are the same (na), but next 2 characters (na vs. ba) show the answer is $>$.  

Sorting based on length $2^i$ prefixes then takes only $O(n \log n)$ time. Since $i$ ranges up to $\log_2 n$, then overall time is $O(n \log^2 n)$. 
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**Example**

*naban* and *nanaban* are adjacent suffixes in the suffix array.

Their common prefix length is 2. This information can easily be construct along with the construction of the suffix array itself.
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**Faster Algorithm**

Getting down to $O(n)$ running time is a bit of a pain, but $O(n \log n)$ isn’t so bad.

We can “bucket sort” each step in $O(n)$ time if we have an appropriate mapping of the length $2^{i-1}$ substrings to integers $\{0, \ldots, n-1\}$. 
Knuth-Morris-Pratt

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Given a **source string** $s$ and a **pattern string** $p$, does $p$ appear as a substring of $a$?

More generally, record all positions $i$ such that $p$ appears as a substring of $a$ starting at position $i$.

**Example**

$s = \text{find} \text{matching} \text{matches}$

$p = \text{match}$

Then $p$ appears as a substring of $s$ at indices 4 and 12 (highlighted).
An obvious algorithm is to try all locations of $s$ and linearly scan to see if $p$ matches there.

Can take $\Omega(|s| \cdot |p|)$ time. The Knuth-Morris-Pratt (KMP) algorithm only takes $O(|s| + |p|)$ time!
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**Main Idea:** For each index $i$ into $p$, let $\pi[i]$ denote the length of the longest proper suffix of $p_0p_1\ldots p_i$ that is also a prefix of $p$. 
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**Main Idea:** For each index $i$ into $p$, let $\pi[i]$ denote the length of the longest proper suffix of $p_0p_1\ldots p_i$ that is also a prefix of $p$.

**Confusing? Example!**

$p = \text{acabaca}$
The longest proper suffix of acabac that is also a prefix is ac.

$\pi[] = \{0, 0, 1, 0, 1, 2, 3\}$;
Slide the pattern $p$ "over" $s$.

acabaca
acacabacabaca
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acabaca
acacabacabaca

Match as many symbols as possible

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acacabacabaca
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acabaca
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When stuck, slide the pattern to the next partial match.

acabaca
acacabacabaca

Distance to slide encoded by prefix table $\pi$. 
Continue matching

acabaca
acacabacabaca
Continue matching

    acabaca
acacabacabaca

Found a match, record it! Slide pattern over to the next partial match.

    acabaca
acacabacabaca
Continue matching

    acabaca
acacabaca acabaca
acacabaca acabaca

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    acabaca
acabaca acabaca
acacabacabaca acabaca
acacabaca acabaca

Continue matching

    acabaca
acabaca acabaca
acacabaca acabaca
acacabaca acabaca

Another match, record it!
Slide pattern over to next partial match.

\texttt{acabaca}
\texttt{acacabacabaca}
Slide pattern over to next partial match.

 acabaca
acacabacabaca

Quit, the pattern is past the end of the string.
Slide pattern over to next partial match.

\texttt{acabaca}
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Runs in $O(|s| + |p|)$ time because each step increases the “matched” pointer or slides the pattern over. Each slide takes $O(1)$ time using the $\pi$ values.
void kmp(const string& s, const string& p) {
    vector<int> pi;
    compute_prefix(p, pi); // next two slides :)

    // invariant: at the start of each iteration hit is the
    // length of the longest *proper* prefix of p[] that
    // matches the suffix of s[0...(i-1)]
    for (int i = 0, hit = 0; i < s.length(); ++i) {
        // slide the window until a hit (or slid past)
        while (hit > -1 && p[hit] != s[i]) hit = pi[hit];

        // or do whatever to process the match, just
        // make sure hit is incremented for sure and is
        // shifted back to p[hit] if there is a match
        if (++hit == p.length()) {
            cout << "Match:" << i << endl;
            hit = pi[hit];
        }
    }
}
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$p = \text{bananaban}$

Note, a suffix of $\pi[i]$ that is also a prefix of $p$ comes from a suffix of $\pi[i - 1]$ that is a prefix of $p$.

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So,

- $\pi[i]$ is just $\pi[i - 1]$ if $s[i] == s[\pi[i - 1]]$.
- Otherwise, check $s[i] == s[\pi[\pi[i - 1]]]$ and so on.
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- Otherwise, check $s[i] == s[\pi[\pi[i-1]]]$ and so on.

Overall idea: slide the pattern over itself!

$\text{acabaca}$

$\text{acabaca}$
```cpp
void compute_prefix(const string& p, vector<int>& pi) {
    pi.resize(p.length()+1);
    pi[0] = -1;

    for (int i = 0; i < p.length(); ++i) {
        // start with the shift from the previous character
        pi[i+1] = pi[i];

        // slide the window until the next character matches
        while (pi[i+1] > -1 && p[pi[i+1]] != p[i])
            pi[i+1] = pi[pi[i+1]];

        // we matched a character or slid back to index -1
        // in either case, increment
        ++pi[i+1];
    }
}
```
Missing Topics

- Tries (presented later as a CMPUT 403 project topic)
- Suffix Trees
- Manachar’s Algorithm: find all maximal palindromes in linear time.

Next Week

Bipartite graphs: recognition, matching, and edge colouring.
Starting Question: How can you find the longest substring in common with 2 strings $s$, $t$?

Concatenate $s \cdot t$ and form a suffix array. Find the largest LCP $[i]$ value where $i, i+1$ come from different strings $s, t$, continued next slide.
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*continued next slide*
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For indices $i, j$ into the suffix array, we can compute the number of distinct strings $s^i$ represented by these indices.
Open Kattis - lifeforms

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Examine “minimal” pairs $i, j$ (no “smaller” pair for $> k / 2$ lifeforms).
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Use a heap to hold the $LCP[\ell]$ values for $i \leq \ell < j$. Pop the min if it is irrelevant (i.e. $\ell < i$).
Can identify all occurrences of the bug in a line in $O(s)$ (after building the prefix table for the bug). But what about when a bug is removed? It may introduce a new bug! Can we somehow “resume” the KMP process after removing a bug?

One a bug is removed, rematch the longest possible prefix from the previous unremoved character to resume KMP. Can do in $O(1)$ time if we just remember the longest match at each character. Should also keep track of the next and previous unremoved character for each letter to “jump” the gaps in $O(1)$ time while scanning.
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Open Kattis - chasingsubs

Idea: use KMP but “construct” the permutation on the fly. Still build \( \pi \) for the pattern, but also \( \text{prev}[i] \) mapping an index \( i \) to the previous occurrence of the same letter in the pattern. In the KMP matching stage (i.e. sliding the pattern over the text), also keep track of the permutation of the letter so far. When we trying to match \( p[i] \) to \( s[j] \) when sliding the pattern, if \( \text{prev}[i] \) is defined then matched ensure \( p[\text{prev}[i]] = s[j] \). Otherwise, define the encryption permutation to send \( p[i] \) to \( s[j] \). When sliding the pattern because of “no match”, remove rules from the encryption permutation as you slide past characters. There are some details to consider here, but it can be done in linear time.
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