Weighted Graph Algorithms
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Programming Club Meeting
Weighted Graphs

```cpp
struct Edge {
    int u, v;
    int weight; // can be a double

    Edge (int uu = 0, int vv = 0, int ww = 0) :
        u(uu), v(vv), w(ww) {}
    bool operator < (const Edge& rhs) const {
        return weight < rhs.weight;
    }
};

typedef vector<vector<Edge>> weighted_graph;
weighted_graph g(n); // create a graph with n nodes

// read and add an edge
Edge e;
cin >> e.u >> e.v >> e.weight;
g[e.u].push_back(e);
```
Dijkstra’s Algorithm

Find the minimum-weight path from $s$ to every other vertex.

Assumption: $w(u, v) \geq 0$ for all edges $uv$. 
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Light a fire at s. The fire takes \( w(s, v) \) seconds to reach every neighbour \( v \) of s.

When a fire reaches a vertex \( v \), if the vertex has not yet been burned then the fire spreads down each edge exiting \( v \).
Use the Edge struct to model active fires spreading across edges. 
Edge.weight is the time when the fire reaches Edge.v.
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```cpp
weighted_graph g; // assume already populated

priority_queue pq<Edge>; // active fires
vector<Edge> path(g.size(), Edge(-1, -1, -1));

pq.push(Edge(s, s, 0)); // a fire starts on s at time 0

while (!pq.empty()) {
    Edge curr = pq.top();
    pq.pop();
    if (path[curr.v].u != -1) continue; // already burned
    path[curr.v] = curr;
    for (auto& succ : g[curr.v])
        pq.push(Edge(succ.u, succ.v,
                      succ.weight + curr.weight));
}
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```

Now `path[v].u` is the vertex prior to `v` on a min-weight `s`—`v` path and `path[v].weight` is the weight of this path.
Problem
A c++ priority_queue is a max-heap, meaning it will return the largest item in the heap.

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A `cpp priority_queue` is a max-heap, meaning it will return the largest item in the heap.

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Running Time
\( O(m \log m) \) where \( m = \# \text{ edges} \).

An edge is burned at most once, and it takes \( O(\log m) \) time to push and pop.
Handling Negative Weights

Things are tricker with minimum-weight cycles.

- A walk from $s$ to $t$ could run around a negative weight cycle as long as it wants, so there is no minimum-weight walk.
- If we insist on not repeating a vertex, it is NP-hard to find the minimum-weight path.

Shortest paths can still be found if no negative weight cycles exist.
Adjacency Matrix Representation

It might make more sense to use \( \infty \) on the diagonal. It depends on the application.
Floyd-Warshall

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Let $G$ be represented with an adjacency matrix with 0s on the diagonal.
/* Assumes \( g[u][v] \) is initially the cost of the edge \((u,v)\), 
INFINITY (i.e. some big number) if no such edge. 
Also need \( g[v][v] = 0 \) for all \( v \). */
for (int \( k = 0 \); \( k < n \); \( k++ \))
for (int \( u = 0 \); \( u < n \); \( u++ \))
  for (int \( v = 0 \); \( v < n \); \( v++ \))
    \( g[u][v] = \min (g[u][v], g[u][k] + g[k][v]) \);
/* Assumes $g[u][v]$ is initially the cost of the edge $(u,v)$, INFINITY (i.e. some big number) if no such edge. Also need $g[v][v] = 0$ for all $v$. */
for (int $k = 0$; $k < n$; $k++$)
  for (int $u = 0$; $u < n$; $u++$)
    for (int $v = 0$; $v < n$; $v++$)
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If there is a negative weight cycle $C$, $g[v][v] < 0$ for some $v \in C$.

Otherwise, $g[u][v]$ is the weight of the min-weight $u - v$ path for any $u, v \in V$. 
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   Also need g[v][v] = 0 for all v. */
for (int k = 0; k < n; k++)
    for (int u = 0; u < n; u++)
        for (int v = 0; v < n; v++)
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**Invariant**
After iteration \( k \), \( g[u][v] \) is the minimum possible weight of a \( u - v \) path using only uses 0, \ldots, \( k \) as intermediate vertices.
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path using only uses 0, . . . , k as intermediate vertices.

Running Time: O(n^3).
Bellman-Ford

A (potentially) faster way to handle negative-weight edges.

Let $s \in V$, let $bf[s] = 0$ and $bf[v] = \infty$ for all $v \neq s$.

The values $bf[v]$ represent the shortest $s - v$ path “seen so far”.
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Try to find an edge $(u, v)$ with $bf[u] + weight(u, v) < bf[v]$ to find a shorter path to $v$.

Iterate until no more changes to $bf[]$. 
int bf[MAXN]; // MAXN = max # nodes possible
vector<Edge> edges; // list of all edges

for (int v = 0; v < n; ++v)
    bf[v] = (v == s ? 0 : INFINITY);

for (int iter = 0; iter < n; ++iter)
    for (auto& e : edges)
        bf[e.v] = min(bf[e.v], bf[e.u] + e.weight);
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**Running Time:** \( O(n \cdot m) \)
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**Running Time:** $O(n \cdot m)$

**Loop Invariants:**
1. If $bf[v] \neq \infty$, it is the length of some $s - v$ path.
2. After $k$ iterations, $bf[v] \leq$ shortest path that uses $\leq k$ steps.
All-pairs shortest paths with negative-weight edges

Initialize $bf[v] = 0$, $\forall v \in V$ and run the Bellman-Ford algorithm.

Note: $bf[u] + weight(u, v) < bf[v] < 0$ for some edge $uv$ iff $G$ has a negative-weight cycle.
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If there is no negative weight cycle:

$$bf[u] + weight(u, v) - bf[v] \geq 0, \forall uv \in E$$
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Find all-pairs shortest paths by running Dijkstra’s from each \( v \in V \), but use edge weights \( bf[u] + weight(u, v) - bf[v] \).
All-pairs shortest paths with negative-weight edges

Initialize $bf[v] = 0, \forall v \in V$ and run the Bellman-Ford algorithm.

Note: $bf[u] + \text{weight}(u, v) < bf[v] < 0$ for some edge $uv$ iff $G$ has a negative-weight cycle.

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Running time: $O(nm \log m)$. Better than $O(n^3)$ in sparse graphs.
Minimum Spanning Tree

Let $G = (V, E)$ be an undirected, connected, and weighted graph.

**Spanning Tree**
A subset of edges $T \subseteq E$ forming a connected tree.

Find the spanning tree with minimum total edge weight.
Kruskal’s Algorithm

- Sort the edges $e_1, \ldots, e_m$ by weight.
- $T = \emptyset$
- For each $e_i = (u, v)$ in this order, if $u$ is not connected to $v$ in $T$ then add $e_i$ to $T$. 

Exercise
- The final $T$ will always be connected (assuming $G$ is connected).
- There are no cycles in $T$.
- So, $T$ is a spanning tree.
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Greedy algorithms like this often fail to find optimum solutions. Why does it work in this case?
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Lemma (good coffee shop math)

A spanning tree $T$ is a minimum-weight spanning tree if and only if for every cycle $C$, some heaviest edge of $C$ is not in $T$. 

So, when the algorithm considers the heaviest edge $e_i$ of some cycle, its endpoints are already connected so $e_i$ will not be added!
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So, when the algorithm considers the heaviest edge $e_i$ of some cycle, its endpoints are already connected so $e_i$ will not be added!
To **efficiently** detect if $T$ connects the endpoints of an edge $e_i$, we use the **union-find** data structure.

It represents a connected component in $T$ by a directed tree, the root is the *representative* of the tree.

The endpoints of $e_i$ have the same representative if and only if they are already connected.
Before adding any edges (i.e. \( T = \emptyset \)): 

![Diagram of connected components]

- a
- b
- c
- d
- e
- f

To merge two components, just point one representative at the other.
Before adding any edges (i.e. $T = \emptyset$):

![Diagram showing separate components labeled a, b, c, d, e, and f.]

To merge two components, just point one representative at the other.

![Diagram showing an edge connecting representatives to merge components.]
Crawl up the tree to find representatives.

Path Compression
To speed up future calculations, also point all nodes seen to the rep.
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int uf[MAXN]; //union find pointers
int find(int u) {
    if (uf[u] != u) uf[u] = find(uf[u]);
    return uf[u];
}

bool merge(int u, int v) {
    u = find(u); v = find(v);
    uf[u] = v;
    return u == v; //true iff this merged different components
}
...
for (int v = 0; v < n; ++i) uf[v] = v; //initialization

//suppose edges are stored in vector<Edge> edges;
sort(edges.begin(), edges.end());

int mst = 0;
for (auto & e : edges)
    if (merge(e.u, e.v))
        mst += e.weight;
Running time:

- $O(m \log m)$ to sort the edges
- $m$ union-find “merges” has total running time $O(m \log m)$

Overall: $O(m \log m)$. 
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Union Find by Rank (in textbook)
$k$ merge calls has running time $O(\alpha(k) \cdot k)$ where $\alpha(k)$ is a crazy slow function.

$\alpha(k) \leq 5$ for $k \leq 100^{100}$

Though, $\alpha(k) \to \infty$ so it’s technically not a constant.
Open Kattis - borg

https://open.kattis.com/problems/borg
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The path of edges taken looks like a tree and the nodes are either S or A nodes.

Compute the shortest path distance between any two of these characters: \(O(a \cdot x \cdot y)\) time using a BFS from each of the \(a\) aliens.

Then compute the minimum spanning tree of the shortest-path graph in \(O(a \log a)\) time.

**Overall**: \(O(a \cdot x \cdot y)\) time
Open Kattis - allpairspath

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Recall the running times. What was the fastest one for this task?

Not Bellman-Ford, not Floyd-Warshall, not Dijkstra’s.

The Dijkstra + Bellman-Ford hybrid!

Precomputes all-pairs shortest path distances in $O(nm \log m)$ time, each query can be answered in $O(1)$ time.
Compute any spanning tree $T$, first by processing the red edges. Return no if no spanning tree or if $> k$ blue edges are used. Re-run Kruskal’s algorithm by first adding blue edges of $T$, then the remaining edges until exactly $k$ are added, then adding red.

Correctness Idea: The first pass identifies the minimum number of blue edges required. Let $T^*$ be a feasible solution. For any blue $e \in T - T^*$, there is some blue $f \in T^* - T$ with $T^* - f + e, T - e + f$ being spanning trees. So $T^* - f + e$ is another feasible solution agreeing more with $T$ on blue edges. Iterating shows $B_T$ can be extended to a feasible solution.
Open Kattis - redblueetree

https://open.kattis.com/problems/redblueetree

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