Weighted Graph Algorithms

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Programming Club Meeting

Weighted Graphs

```
struct Edge {
  int u, v;
  int weight; //can be a double

  Edge (int uu = 0, int vv = 0, int ww = 0)
    : u(uu), v(vv), w(ww) {}
  bool operator<(const Edge& rhs) const {
    return weight < rhs.weight;
  }
};

typedef vector<vector<Edge>> weighted_graph;
weighted_graph g(n); //create a graph with n nodes

//read and add an edge
Edge e;
cin >> e.u >> e.v >> e.weight;
g[e.u].push_back(e);
```

Dijkstra's Algorithm

Find the minimum-weight path from s to every other vertex.

Assumption: $w(u, v) \geq 0$ for all edges uv.

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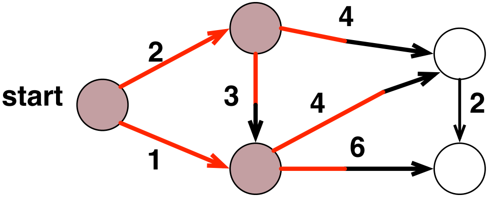
Light a fire at s. The fire takes $w(s, v)$ seconds to reach every neighbour v of s.



When a fire reaches a vertex v, if the vertex has not yet been *burned* then the fire spreads down each edge exiting v.

Use the `Edge` struct to model active fires spreading across edges.
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```
weighted_graph g; //assume already populated

priority_queue pq<Edge>; //active fires
vector<Edge> path(g.size(), Edge(-1, -1, -1));

pq.push(Edge(s, s, 0)); //a fire starts on s a time 0

while (!pq.empty()) {
   Edge curr = pq.top();
   pq.pop();
   if (path[curr.v].u != -1) continue; //already burned
   path[curr.v] = curr;
   for (auto& succ : g[curr.v])
      pq.push(Edge(succ.u, succ.v,
                   succ.weight + curr.weight));
}
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Now $path[v].u$ is the vertex prior to v on a min-weight $s - v$ path and $path[v].weight$ is the weight of this path.

Problem

A `c++` `priority_queue` is a max-heap, meaning it will return the largest item in the heap.

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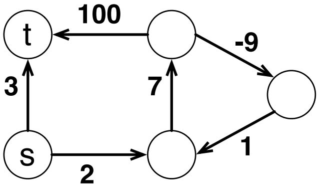
Running Time

$O(m \log m)$ where $m = \#$ edges.

An edge is burned at most once, and it takes $O(\log m)$ time to push and pop.

Handling Negative Weights

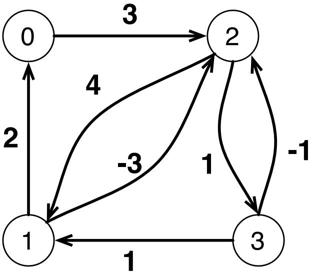
Things are tricker with minimum-weight cycles.



- A walk from s to t could run around a negative weight cycle as long as it wants, so there is no *minimum-weight walk*.
- If we insist on not repeating a vertex, it is NP-hard to find the minimum-weight path.

Shortest paths can still be found if no negative weight cycles exist.

Adjacency Matrix Representation



$$A = \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & -3 & \infty \\ \infty & 4 & 0 & 1 \\ \infty & 1 & -1 & 0 \end{bmatrix} \quad (\infty \equiv \text{no edge})$$

It might make more sense to use ∞ on the diagonal. It depends on the application.

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Let G be represented with an adjacency matrix with 0s on the diagonal.

```
/* Assumes g[u][v] is initially the cost of the edge (u,v),
   INFINITY (i.e. some big number) if no such edge.
   Also need g[v][v] = 0 for all v. */
for (int k = 0; k < n; k++)
   for (int u = 0; u < n; u++)
      for (int v = 0; v < n; v++)
         g[u][v] = min(g[u][v], g[u][k] + g[k][v]);
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If there is a negative weight cycle C, $g[v][v] < 0$ for some $v \in C$.

Otherwise, $g[u][v]$ is the weight of the min-weight $u - v$ path for any $u, v \in V$.

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Invariant
After iteration k, $g[u][v]$ is the minimum possible weight of a $u - v$ path using only uses $0, \ldots, k$ as intermediate vertices.

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Running Time: $O(n^3)$.

Bellman-Ford

A (potentially) faster way to handle negative-weight edges.

Let $s \in V$, let $bf[s] = 0$ and $bf[v] = \infty$ for all $v \neq s$.

The values $bf[v]$ represent the shortest $s - v$ path “seen so far”.

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Try to find an edge (u, v) with $bf[u] + weight(u, v) < bf[v]$ to find a shorter path to v.

Iterate until no more changes to $bf[]$.

```
int bf[MAXN]; //MAXN = max # nodes possible
vector<Edge> edges; //list of all edges
...
for (int v = 0; v < n; ++v)
    bf[v] = (v == s ? 0 : INFINITY);

for (int iter = 0; iter < n; ++iter)
    for (auto& e : edges)
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Running Time: $O(n \cdot m)$

Loop Invariants:

1. If $bf[v] \neq \infty$, it is the length of some $s - v$ path.
2. After k iterations, $bf[v] \leq$ shortest path that uses $\leq k$ steps.

All-pairs shortest paths with negative-weight edges

Initialize $bf[v] = 0, \forall v \in V$ and run the Bellman-Ford algorithm.

Note: $bf[u] + weight(u, v) < bf[v] < 0$ for some edge uv iff G has a negative-weight cycle.

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If there is no negative weight cycle:

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Find all-pairs shortest paths by running Dijkstra's from each $v \in V$, but use edge weights $bf[u] + weight(u, v) - bf[v]$.

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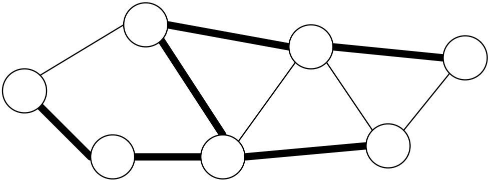
Running time: $O(nm \log m)$. Better than $O(n^3)$ in sparse graphs.

Minimum Spanning Tree

Let $G = (V, E)$ be an undirected, connected, and weighted graph.

Spanning Tree

A subset of edges $T \subseteq E$ forming a connected tree.



Find the spanning tree with minimum total edge weight.

Kruskal's Algorithm

- Sort the edges $e_1, \ldots, e_m$ by weight.
- $T = \emptyset$
- For each $e_i = (u, v)$ in this order, if u is not connected to v in T then add e_i to T.

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Exercise

- The final T will always be connected (assuming G is connected).
- There are no cycles in T.

So, T is a spanning tree.

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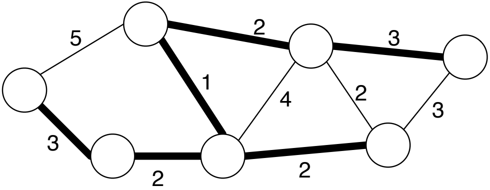
Lemma (good coffee shop math)

A spanning tree T is a minimum-weight spanning tree if and only if for every cycle C, some heaviest edge of C is not in T.



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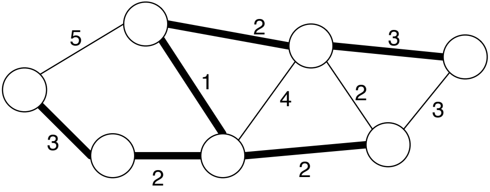
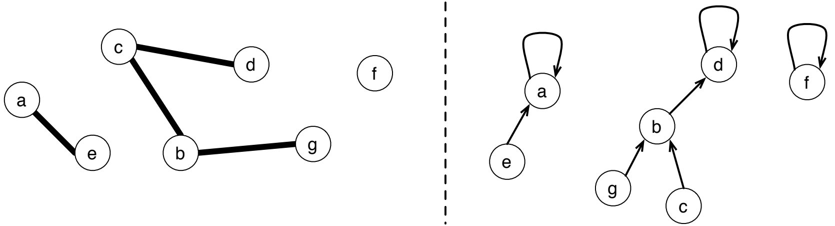


So, when the algorithm considers the heaviest edge e_i of some cycle, its endpoints are already connected so e_i will not be added!

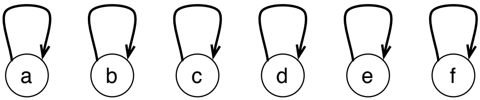
To **efficiently** detect if T connects the endpoints of an edge e_i, we use the union-find data structure.

It represents a connected component in T by a directed tree, the root is the *representative* of the tree.



The endpoints of e_i have the same representative if and only if they are already connected.

Before adding any edges (i.e. $T = \emptyset$):



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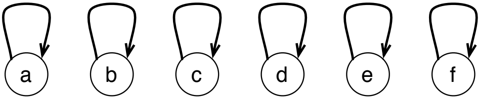
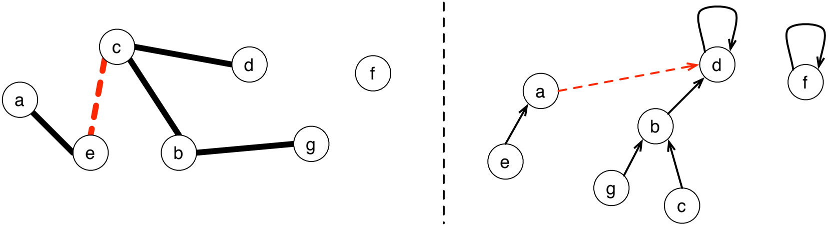
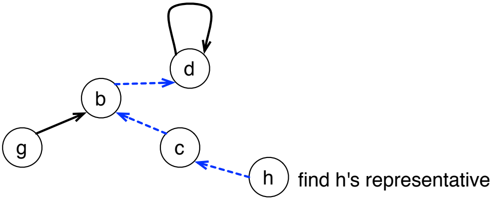


To merge two components, just point one representative at the other.



Crawl up the tree to find representatives.



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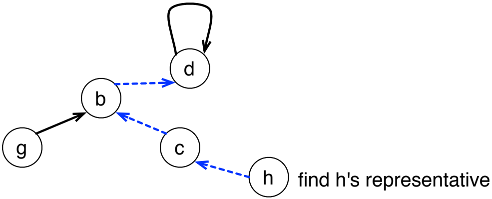
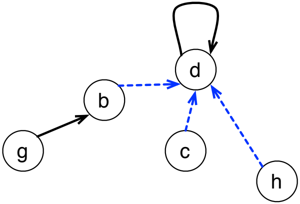


Path Compression

To speed up future calculations, also point all nodes seen to the rep.



```
int uf[MAXN]; //union find pointers
int find(int u) {
  if (uf[u] != u) uf[u] = find(uf[u]);
  return uf[u];
}
bool merge(int u, int v) {
  u = find(u); v = find(v);
  uf[u] = v;
  return u == v; //true iff this merged different components
}
...
for (int v = 0; v < n; ++i) uf[v] = v; //initialization

//suppose edges are stored in vector<Edge> edges;
sort(edges.begin(), edges.end());

int mst = 0;
for (auto& e : edges)
  if (merge(e.u, e.v))
    mst += e.weight;
```

Running time:

- $O(m \log m)$ to sort the edges
- m union-find “merges” has **total** running time $O(m \log m)$

Overall: $O(m \log m)$.

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Union Find by Rank (**in textbook**)
k merge calls has running time $O(\alpha(k) \cdot k)$ where $\alpha(k)$ is a crazy slow function.

$\alpha(k) \leq 5$ for $k \leq 100^{100^{100}}$

Though, $\alpha(k) \rightarrow \infty$ so it's technically not a constant.

Open Kattis - `borg`

`https://open.kattis.com/problems/borg`

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The path of edges taken looks like a tree and the nodes are either `S` or `A` nodes.

Compute the shortest path distance between any two of these characters: $O(a \cdot x \cdot y)$ time using a BFS from each of the a aliens.

Then compute the minimum spanning tree of the shortest-path graph in $O(a \log a)$ time.

Overall: $O(a \cdot x \cdot y)$ time

Open Kattis - allpairspath

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Recall the running times. What was the fastest one for this task?

Not Bellman-Ford, not Floyd-Warshall, not Dijkstra's.

The Dijkstra + Bellman-Ford hybrid!

Precomputes all-pairs shortest path distances in $O(nm \log m)$ time, each query can be answered in $O(1)$ time.

Open Kattis - `redbluetree`

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Compute any spanning tree T, first by processing the red edges. Return `no` if no spanning tree or if $> k$ blue edges are used.

Re-run Kruskal's algorithm by first adding blue edges of T, then the remaining edges until exactly k are added, then adding red.

Correctness Idea: The first pass identifies the minimum number of blue edges required.

Let T^* be a feasible solution. For any blue $e \in T - T^*$, there is some blue $f \in T^* - T$ with $T^* - f + e$, $T - e + f$ being spanning trees. So $T^* - f + e$ is another feasible solution agreeing more with T on blue edges. Iterating shows B_T can be extended to a feasible solution.