Unweighted Graphs & Algorithms

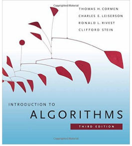
Zachary Friggstad

Programming Club Meeting

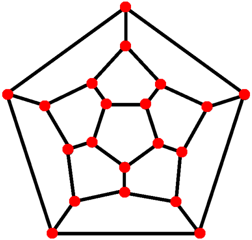
References

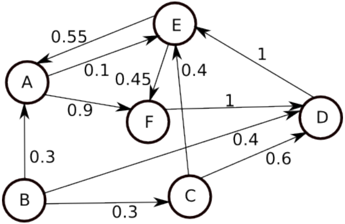
Chapter 4: Graph (Section 4.2)

Chapter 22: Elementary Graph Algorithms



Graphs





Features: vertices/nodes/dots and edges/links/lines between vertices.

Sometimes there are labels or numeric values associated with the items in the graph. The edges may or may not have directions.

We will discuss over a couple of meetings how to model graphs and some graph algorithms.

Unweighted Graphs

Note

Many code snippets here use `C++11` features. Compile with the flag `-std=c++11` if using `g++`.

Throughout, $n = \#$ vertices, $m = \#$ edges.

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Adjacency List Representation of a Graph

```
//without c++11 you may need to add a space between >>
typedef vector<vector<int>> graph;
...
graph g(n);          //create a graph with n vertices
g[u].push_back(v); //add v as a neighbour of u
```

For undirected graphs, just add both directions of an edge (u, v).
Requires $\Theta(n + m)$ space.

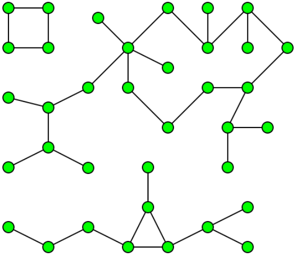
Problem: Reachability

Give a vertex *v* in a graph, find all vertices *u* that *v* can reach.

That is, we can reach *u* by following a sequence of edges (in the right direction, if the graph is directed).

Example

The top, left vertex can only reach the other vertices in the "square".



Depth-First Search

Find all vertices reachable from vertex v.

```
//the vertices that are reached in the search
vector<bool> reached(n, false);
graph g;

void dfs(int u) {
   if (!reached[u]) {
      reached[u] = true;
      for (auto w : g[u]) dfs(w);
   }
}
...
dfs(v);
```

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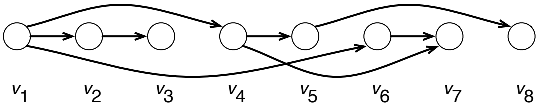
dfs(v);
```

If we record the vertex that discovered u, we can reconstruct paths.

Runs in $O(n + m)$ time.

Other Applications of DFS: Topological Sorting

Order the vertices so all edges point left-to-right.



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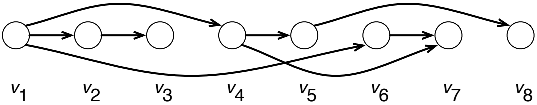

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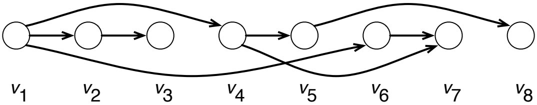

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- Begin a DFS. Just before returning from a recursive call (i.e. just after the `for` loop) `push_back` the vertex u to the end of a vector.

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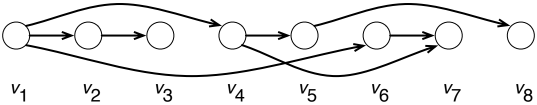


Impossible to do if there is a cycle. Otherwise, the following works.

- Begin a DFS. Just before returning from a recursive call (i.e. just after the `for` loop) `push_back` the vertex u to the end of a vector.
- Repeat, starting with an unvisited vertex each time, until all vertices are visited.

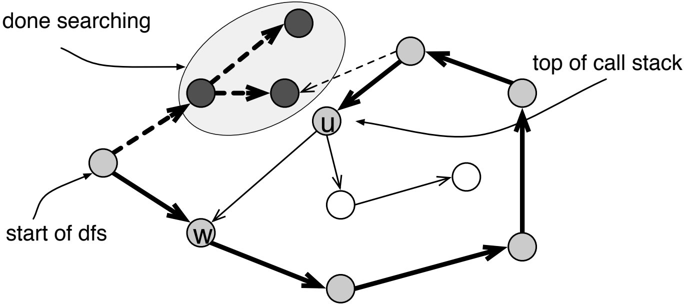
```
vector<int> order; //initially empty

void topo_sort(int u) {
   if (!reached[u]) {
      reached[u] = true;
      for (auto w : g[u]) topo_sort(w);
      order.push_back(u);
   }
}

...

for (int u = 0; u < n; u++)
   if (!reached[u])
      topo_sort(u);
reverse(order.begin(), order.end()); //#include <algorithm>
```

If u is ordered after w for some edge (u, w), it must be that the recursive call with w was on the call stack when u was being processed. (**Why?**)



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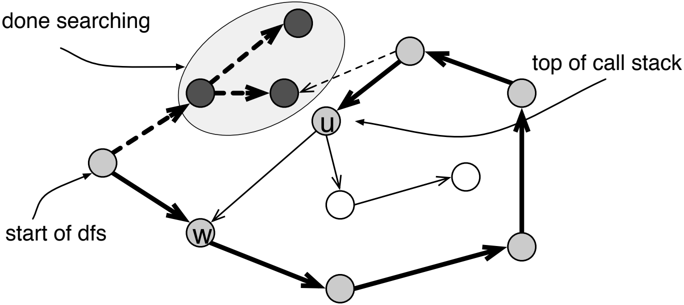

If w is on the call stack when u is being processed, there is a path from w to u. Completing this path with the edge (u, w) yields a cycle.

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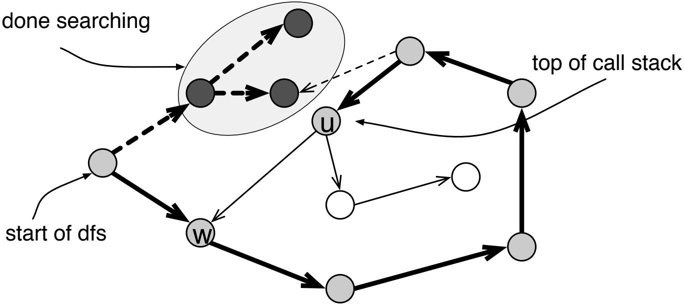


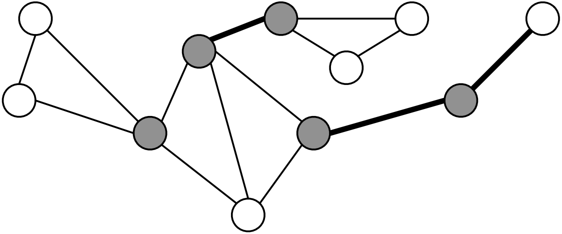
If w is on the call stack when u is being processed, there is a path from w to u. Completing this path with the edge (u, w) yields a cycle.

Thus

If the graph has no cycles, this will topologically sort all vertices.

Articulation Points & Bridges

An articulation point in an undirected, connected graph is a vertex whose removal leaves a disconnected graph.



A bridge is an edge whose removal leaves a disconnected graph.

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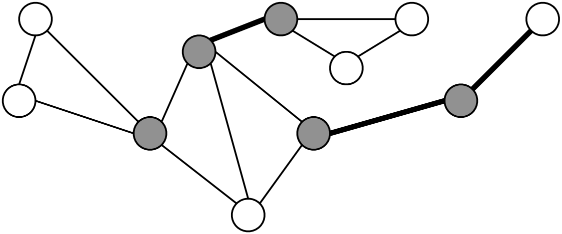
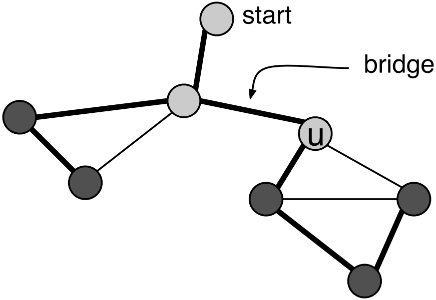


A bridge is an edge whose removal leaves a disconnected graph.

Can find all bridges and articulation points in $O(n + m)$ time via DFS.

A bridge will always be a tree edge in a DFS (actually, in any spanning tree): one that is expanded along in the search.



Picture: no edge of a descendent of *u* in the search reached a non-descendent. So the parent edge of *u* is a bridge.

Run a DFS, record the order the vertices were discovered.

Return the earliest discovery time of any vertex adjacent to a descendant of u. This indicates if some descendant is adjacent to a non-descendant.

```
vector<int> found(n, -1); // discovery time
int cnt = 0;

int bridges(int u, int p) {
   if (found[u] != -1) return found[u];
   int mn = found[u] = cnt++; //record u's discovery time
   for (auto w : g[u])
     mn = min(mn, bridges(w, u));
   if (mn == found[u] && p != -2)
      // (p, u) is a bridge, process it how you want
   return mn;
}
...
bridges(0, -2); //start the search from any vertex
```

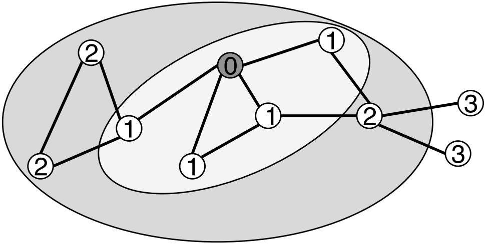
Other DFS Applications

- Find all articulation points in a graph (good exercise).
- Find the *strongly connected components* of a directed graph.
- Compute pre/post order traversals of a tree.
- Deciding if a graph is bipartite.
- Simple code for augmenting a bipartite matching (later lecture).

All of these can be implemented to run in $O(n + m)$ time.

Breadth-First Search

A *breadth-first search* will explore the vertices in increasing order of their shortest path distance from the start vertex.



- Load up the start vertex in a queue q.
- While q is not empty, extract the front vertex and add all of its unvisited neighbours to the back of q.

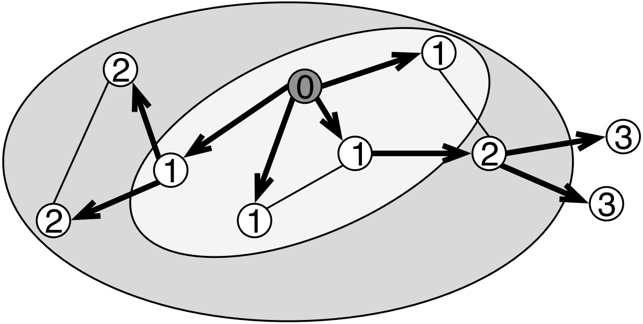
```
queue<int> q; //#include <queue>
vector<int> prev(n, -1);

q.push(v);    //v is the start vertex in the search
prev[v] = -2; //signals "root of search"

while (!q.empty()) {
  int curr = q.front();
  q.pop();
  for (auto succ : g[curr])
    if (prev[succ] == -1) {
      prev[succ] = curr;
      q.push(succ);
    }
}
```

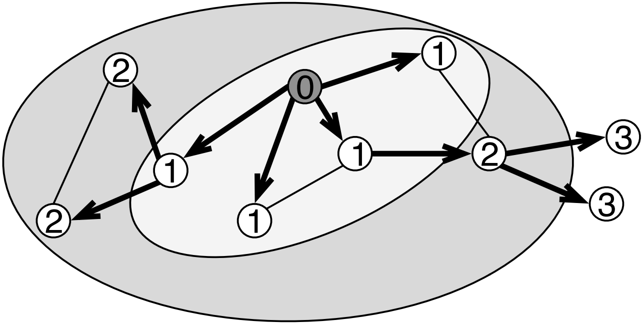
Now $prev[u]$ for $u \neq v$ is the vertex prior to u on a shortest $v - u$ path.

Also runs in $O(n + m)$ time.



A thick arrow from u to w indicates $prev[w] = u$.

The unique path using thick arrows from the start vertex (**dark**) to any vertex is a shortest path in the graph.



A thick arrow from *u* to *w* indicates $prev[w] = u$.

The unique path using thick arrows from the start vertex (**dark**) to any vertex is a shortest path in the graph.

Though we illustrated with an undirected graph, the same algorithm also finds shortest paths in directed graphs.

To Come...

Next week

Algorithms for weighted graphs.

- Dijkstra's algorithm for shortest paths.
- Floyd-Warshall for all-pairs shortest paths.
- Bellmand-Ford: handling negative weight cycles.
- Minimum Spanning Trees: Kruskal's Algorithm.
- Eulerian Circuits

Later in the term

- Bipartite matching: unweighted.
- Network flow: max-flow/min-cut.

Example Problem: Kattis - `coast`

`https://open.kattis.com/problems/coast`

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- Run a dfs from an "outside" vertex.
- Return the number of coastlines seen from each recursive call.
- Do not continue the search if you try to cross to a land vertex, just return 1.
- **Tip**: Pad the grid with water tiles on each edge, so you know $(0, 0)$ is a water tile and you only have to run 1 dfs.

Running time: $O(N \cdot M)$, which is linear in the size of the grid.

Example Problem: Kattis - `eulerianpath`

`https://open.kattis.com/problems/eulerianpath`

Let δ_v be the difference between the outdegree and indegree of v.

- Start a search at some s with $\delta_s = +1$ (pick any vertex s if there is no such vertex).
- When processing v in the search, do the following. While there is an unused outgoing edge vw, recursively search from w.
- When all edges exiting v are used, append v to the tour and return.

Check that the tour contains $m + 1$ nodes (so all edges are traversed). If so, reverse it to get an Eulerian tour starting at s.

Running time: $O(n + m)$.

Tip: Emulate the DFS with a stack if the recursion can go too deep.

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Running time $O(2^{2n})$.
If in step 1 we only keep *maximal* independent sets (i.e. not contained in a larger ind. set), the running time reduces to $O(2^{4n/3})$ because # of maximal independent sets in any graph is $O(2^{n/3})$.