Geometry

Zachary Friggstad

Programming Club Meeting
Points

```cpp
#include <complex>
typedef complex<double> point;

point p(1.0, 5.7);
p.real(); // x component
p.imag(); // y component
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typedef complex<double> point;

point p(1.0, 5.7);
p.real(); // x component
p.imag(); // y component

Helpful operations

point p, q;
p+q; // exactly what you expect
p *= polar(1, theta); // rotate around (0,0) by theta radians
abs(p-q); // distance between p and q
arg(p); // angle from positive x-axis, measured in (-pi, pi]
#include <complex>
typedef complex<double> point;

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One Annoyance: we can’t update the x and y components without creating a new point.
Dot Product

\[ p \circ q = p_x \cdot q_x + p_y \cdot q_y = ||p|| \cdot ||q|| \cdot \cos \theta \]

def dot(point p, point q):
    return real(p*conj(q));
Dot Product

\[ p \circ q = p_x \cdot q_x + p_y \cdot q_y = \|p\| \cdot \|q\| \cdot \cos \theta \]

double dot(point p, point q) {
   return real(p*conj(q));
}

- \( p \circ q = 0 \) if \( p \perp q \) (or one of them is 0)
- \( p \circ p = \|p\|^2 \)
- \( p \circ q > 0 \): \( q \) lies in the same general direction as \( p \)
- \( p \circ q < 0 \): \( q \) lies in the opposite general direction as \( p \)
Cross Product

\[ p \times q = p_x \cdot q_y - p_y \cdot q_x = ||p|| \cdot ||q|| \cdot \sin \theta \]

double cross(point p, point q) { return imag(p*conj(q)); }

positive side (CCW turn from p)

negative side (CW turn from p)
Cross Product

\[ p \times q = p_x \cdot q_y - p_y \cdot q_x = ||p|| \cdot ||q|| \cdot \sin \theta \]

double cross(point p, point q) {
    return imag(p*conj(q));
}

- \( p \times q = 0 \) if \( p \) and \( q \) are collinear with \( 0 \) (or one of them is 0)
- \( p \times q > 0 \): \( \angle(p, q) \) is counterclockwise
- \( p \times q < 0 \): \( \angle(p, q) \) is clockwise
\[ \frac{1}{2} |(p - r) \times (q - r)| \] is the area of the triangle with corners \( p, q, r \).
$\frac{1}{2}|(p - r) \times (q - r)|$ is the area of the triangle with corners $p, q, r$.

**Tip:** $p$ lies on the line passing through points $q$ and $r$ if and only if $(q - p) \times (r - p) = 0$.

**Additionally:** if $p$ lies on this line then it lies between $q$ and $r$ if and only if $(q - p) \circ (r - p) \leq 0$. 
Parametric representation of a line passing through $p$ and $q$: 

$$r(t) = (1 - t) \cdot p + t \cdot q, \quad t \in \mathbb{R}$$

Note $r(0) = p$ and $r(1) = q$. 
Parametric representation of a line passing through $p$ and $q$:

$$ r(t) = (1 - t) \cdot p + t \cdot q, \quad t \in \mathbb{R} $$

Note $r(0) = p$ and $r(1) = q$.

The line segment connecting $p$ and $q$ is parameterized this way, with the restriction $0 \leq t \leq 1$. 
Given two lines passing through \( p, q \) and \( p', q' \) respectively, compute their intersection point (if any).

Solve \((1 - t) \cdot p + t \cdot q = (1 - t')p' + t' \cdot q'\) for \( t, t' \in \mathbb{R} \).
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\[
\begin{bmatrix}
q_x - p_x & p'_x - q'_x \\
q_y - p_y & p'_y - q'_y
\end{bmatrix}
\cdot
\begin{bmatrix}
t \\
t'
\end{bmatrix}
=
\begin{bmatrix}
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Use Cramer’s rule: for \( A \cdot x = b \) we have

\[
x_i = \frac{\det A_i}{\det A}
\]

Where \( A_i \) is the matrix obtained by replacing column \( i \) of \( A \) with \( b \).
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**Tip:** \( \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc \).

\( \det A = 0 \) means the lines are parallel.
Note the $2 \times 2$ determinants can just be replaced by cross products!

```c
// find where lines p1q1 and p2q2 intersect
// assumes p1 != q1, p2 != q2
point intersect(point p1, point q1, point p2, point q2) {
    double den = cross(q1-p1, p2-q2);
    if (fabs(den) < EPS) {
        // parallel lines, do whatever you want
        // they coincide iff fabs(cross(p1-p2, q1-p2)) < 1e-8
    }
    double t = cross(p2-p1, p2-q2)/den;

    return (1-t)*p1 + t*q1;
}
```
Computing the intersection of two line segments.

- If the infinite lines intersect (i.e. not parallel), get values $t, t'$. 
- Ensure $0 \leq t \leq 1$ and $0 \leq t' \leq 1$. 
Computing the intersection of two line segments.

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**If lines are colinear:**

- If \( p = p' \) and \( q = q' \) or vice versa, they overlap.
- Else if a point is in the interior of the other segment, they overlap.
- Else if the segments share a point, they touch only at that point.
- Else they do not touch.
Closest point on a line.

Given a line passing through $p, q$ and another point $c$, find the point $r$ on the line $\overline{pq}$ nearest to $c$. 

Assume $c = (0, 0)$ by shifting if necessary.

Minimize $|| (1 - t) p + t q ||^2$ over $t \in \mathbb{R}$ (avoids the square root).

This is just minimizing some quadratic: $A t^2 + B t + C$.

Tip: the minimum is at $-\frac{B}{A}$. If $A = 0$, then $p = q$. 
Closest point on a line.

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This is just minimizing some quadratic:

\[
A \cdot t^2 + B \cdot t + C
\]

**Tip:** the minimum is at \(-\frac{B}{2A}\). If \( A = 0 \), then \( p = q \).
After working out details...

// return the point on the line pq closest to c
point closest_point(point p, point q, point c) {
    p -= c; q -= c;
    double den = norm(p-q);
    if (fabs(den) < 1e-8) {
        // p=q, do whatever you want
    }
    double t = dot(p, p-q)/den;
    return (1-t)*p + t*q + c;
}
Line-Circle Intersection
A line $\overline{pq}$ and a circle $(r, c)$ (radius/center point) does the line puncture the circle?

Assume $c = 0$ by shifting and solve

$$|| (1 - t) \cdot p + t \cdot q ||^2 = r^2$$

Rearrange:

$$A \cdot t^2 + B \cdot t + C = 0.$$

If $A = 0$ means $p = q$.

Tip: Use the quadratic formula

$$-B \pm \sqrt{\Delta}$$

where

$$\Delta = B^2 - 4AC.$$

$\Delta < 0$: line misses, $\Delta = 0$: line tangent, $\Delta > 0$: line punctures.
Line-Circle Intersection
A line $\overline{pq}$ and a circle $(r, c)$ (radius/center point) does the line puncture the circle?

Assume $c = 0$ by shifting and solve $\|(1 - t) \cdot p + t \cdot q\|^2 = r^2$ for $t$. 
**Line-Circle Intersection**

A line $\overline{pq}$ and a circle $(r, c)$ (radius/center point) does the line puncture the circle?

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// return # of intersections and set parameters
// z1, z2 to these points (if applicable)
int line_circle(point c, double r, point p, point q,
    point& z1, point& z2) {
    p = c; q = c;
    double A = norm(p-q), B = 2*dot(p, q-p), C = norm(p)-r*r;
    double disc = B*B - 4*A*C;

    if (disc + EPS < 0) return 0;

    double rt = sqrt(fabs(disc));
    double t1 = (-B+rt)/(2*A);
    double t2 = (-B-rt)/(2*A);

    z1 = (1-t1)*p + t1*q + c;
    z2 = (1-t2)*p + t2*q + c;
    return (disc < EPS ? 1 : 2);
}
Point \( q \) in Polygon \( P : p_1, \ldots, p_n ? \)

**Intuition:** walk about the polygon with a rope taut with “post” \( q \).

If \( q \) is inside, the rope wraps around once. If \( q \) is outside, the rope doesn’t wrap around.
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If $q$ is inside, the rope wraps around once. If $q$ is outside, the rope doesn’t wrap around.

```cpp
double delta = 0;
for (int i = 0, j = n-1; i < n; j = i++)
    delta += arg((p[i]-q) / (p[j]-q)); // change in angle
return fabs(delta) > 1; // |delta| is 0 or 2*pi
```
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Have to add a bit more to check if $q$ lies on the boundary (cross and dot product test mentioned earlier).
Area of a Polygon $P : p_1, \ldots, p_n$

blue triangle: count the area **positively** (CCW turn about origin)
red triangle: count the area **negatively** (CW turn about origin)
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double area = 0;
for (int $i = 0$, $j = n-1$; $i < n$; $j = i++$)
    //signed area of triangle [origin, p[i-1], p[i]]
    area += cross(p[i], p[j]) * 0.5;
return fabs(area);
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return fabs(area);
```

Points outside are “cancelled”, points inside are counted +1 times (after cancelling). The net sum is the area of $P$. 
Convex Hull

Given points $p_1, \ldots, p_n$, what is their convex hull?

Output: The points on the convex hull in CCW order.
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Convex Hull

The bottom-most point must be on the convex hull. If there are many, choose the leftmost. Suppose this point is \((0, 0)\) by shifting all points if needed.

Sort all other points by their angle with the positive \(x\)-axis. To check if \(p_i < p_j\) according to this order:
Convex Hull

The bottom-most point must be on the convex hull. If there are many, choose the leftmost. Suppose this point is $(0, 0)$ by shifting all points if needed.

Sort all other points by their angle with the positive $x$-axis. To check if $p_i < p_j$ according to this order: check $p_i \times p_j > 0$. 
The origin and the first point after sorting are on the hull for sure.

Build the hull one point at a time: add the points in the sorted order.

If one creates a clockwise turn with the previous two on the hull, then pop the end of the current hull.
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Build up the hull one point at a time: add the points in the sorted order.

If one creates a clockwise turn with the previous two on the hull, then pop the end of the current hull.
bool cmp(point a, point b) { return cross(a, b) > 0; }

point p[MAXN], hull[MAXN];
//Assume p[n-1] is the lowest point (breaking ties by //taking leftmost) and no 3 points are collinear

for (int i = 0; i < n-1; ++i) p[i] = p[n-1];
sort(p, p+n-1, cmp); //sort by angle from x-axis

hull[0] = point(0,0);
int hs = 2;
hull[1] = p[0];

for (int i = 1; i < n-1; ++i) {
    while (cross(hull[hs-1]-hull[hs-2],p[i]-hull[hs-2]) < 0)
        --hs;
    hull[hs++] = p[i];
}
for (int i = 0; i < hs; ++i) hull[i] += p[n-1];

Running time: $O(n \log n)$
Tips

Never use float, not enough precision. Use double.
**Tips**

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```cpp
const double PI = acos(-1.0);
```

Sometimes values that should be equal aren't quite due to floating point errors. Safer comparisons.

```cpp
#define EPS 1e-8

double a, b;
if (fabs(a - b) < EPS) ... // check if a == b
if (a + EPS < b) ... // check if a < b
if (a < b + EPS) ... // check if a <= b
```

Can there be 3 collinear points? Can points be equal? How can you deal with this?
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Can there be 3 collinear points? Can points be equal? How can you deal with this?
Missing Topics
Voronoi diagrams/Delaunay triangulations, 3D convex hull, closest pair of points, furthest pair of points, triangulating a polygon.

Next Time
Number Theory

Some Topics To Come
• Combinatorics and Arithmetic
• String Processing
• Matchings and Network Flow
• Probability Theory
We already talked about how to compute the area of a polygon.

How can you check if it is in CW or CCW order?

Solution: check if the area is positive before returning the absolute value:

> 0 means CCW,
< 0 means CW.
Open Kattis - polygonarea

https://open.kattis.com/problems/polygonarea

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- We already talked about how to compute the area of a polygon.
- How can you check if it is in CW or CCW order?
We already talked about how to compute the area of a polygon. How can you check if it is in CW or CCW order? **Solution:** check if the area is positive before returning the absolute value: $> 0$ means CCW, $< 0$ means CW.
Think: what can we say about the polygon at the tightest squeeze point of the ride.

Will there be a vertex of a polygon at the tightest squeeze?

For each vertex, find the shortest distance to all other line segments on the other polygon (do for both polygons).

Running time: $O(n^2)$. 

https://open.kattis.com/problems/wrapping
Open Kattis - rafting

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Open Kattis - rafting

https://open.kattis.com/problems/wrapping

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Open Kattis - wrapping

https://open.kattis.com/problems/wrapping
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Basic approach:
- The total area of the objects is simple, you know the width and height of each (rectangle area = \( w \cdot h \)).
Open Kattis - wrapping

https://open.kattis.com/problems/wrapping

Basic approach:

- The total area of the objects is simple, you know the width and height of each (rectangle area = \( w \cdot h \)).
- You need the actual corners to compute the convex hull. Rotate the four corners of the axis-parallel rectangles by the given angle: simple using complex points (recall: \( \text{radians} = \frac{\pi}{180} \cdot \text{degrees} \)).
Open Kattis - wrapping

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• Find the convex hull.
Open Kattis - wrapping

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• Find the convex hull.
• Compute the area of the convex hull using the algorithm discussed earlier.

Running time:
Open Kattis - wrapping

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• Find the convex hull.

• Compute the area of the convex hull using the algorithm discussed earlier.

Running time: $O(n \log n)$
Basic approach:

• Consider the line $L$ passing through the given point $p$ and point $y$ (a parameter) on the vertical axis.
• Use the quadratic expression $|| (1 - t) \cdot (0, y) + t \cdot p ||^2 = 1$. Expand and rearrange in terms of $t$.
• The line $L$ is tangent to the circle if the discriminant $\Delta$.
• The discriminant $\Delta$ is a quadratic in $y$, solve for $\Delta = 0$ using the quadratic formula to get the two values of $y$.

Caution: carefully check each step of your calculations!
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Open Kattis - goofy

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