Dynamic Programming

Zachary Friggstad

Programming Club Meeting
Dynamic Programming by Example

**Example**: Matrix Chain Multiplication

Given matrices $M_1, M_2, \ldots, M_n$ where matrix $M_i$ is an $r_i \times c_i$ matrix and $c_i = r_{i+1}$.

**Goal**: Compute $M_1 \cdot M_2 \cdot \ldots \cdot M_n$, an $r_1 \times c_k$ matrix.
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Associativity tells us there are many ways to do this:

\[ A \cdot (B \cdot C) \quad \text{vs} \quad (A \cdot B) \cdot C \]

or

\[ A \cdot (B \cdot (C \cdot D)) \quad \text{vs} \quad (A \cdot B) \cdot (C \cdot D) \quad \text{vs} \quad (A \cdot (B \cdot C)) \cdot D \]
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What is the fastest way.
Dynamic Programming by Example

Say the cost of multiplying an $a \times b$ matrix $S$ with a $b \times c$ matrix $T$ is $abc = \#$ of element-by-element multiplications when computing $S \times T$ naively.
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Example:

- $A : 1 \times 100$
- $B : 100 \times 100$
- $C : 100 \times 100$
- $D : 100 \times 1$

Cost of $(A \cdot (B \cdot C)) \cdot D$ is $100^3 + 100^2 + 100 = 1010100$.

Cost of $(A \cdot B) \cdot (C \cdot D)$ is $2 \cdot 100^2 + 100 = 20100$. 

Dynamic Programming by Example

Recall we want to compute $A_1 \cdot \ldots \cdot A_k$ where $A_i$ is an $r_i \times c_i$ matrix.

What order of multiplication/“parenthesizing” results in the cheapest calculation?
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Let $\text{cost}(i, j)$ be the cheapest way to compute $A_i \cdot \ldots \cdot A_j$. 
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Recall we want to compute $A_1 \cdot \ldots \cdot A_k$ where $A_i$ is an $r_i \times c_i$ matrix.

What order of multiplication/“parenthesizing” results in the cheapest calculation?

**Idea:**
Let $\text{cost}(i, j)$ be the cheapest way to compute $A_i \cdot \ldots \cdot A_j$.

If $i = j$, nothing to compute (answer is 0).

Otherwise “guess” the outermost multiplication:

$$(A_i \cdot \ldots \cdot A_k) \cdot (A_{k+1} \cdot \ldots \cdot A_j)$$

$$\text{cost}(i, j) = \min_{i \leq k \leq j-1} \text{cost}(i, k) + \text{cost}(k + 1, j) + r_i \cdot c_k \cdot c_j.$$
The Recurrence

Concisely,

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\text{cost}(i, j) = \begin{cases} 
0 & \text{if } i = j \\
\min_{i \leq k \leq j-1} \text{cost}(i, k) + \text{cost}(k + 1, j) + r_i \cdot c_k \cdot c_j & \text{if } i < j 
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\end{cases}$$

```c
int cost(int i, int j) {
    if (i == j) return 0; // base case
    int best = INT_MAX;
    for (int k = i; k < j; k++)
        best = min(best,
            cost(i, k) + cost(k+1, j) + r[i]*c[k]*c[j]);
    return best;
}
```
Memoization

This takes exponential time.

Fantastic Idea: Computed each cost\((i, j)\) entry only once!
Memoization

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**Fantastic Idea:** Computed each \( \text{cost}(i,j) \) entry only once!

```c
// table[][] is initialized to contain all -1 entries
int cost(int i, int j) {
    if (table[i][j] == -1) { // first time computing cost(i,j)
        if (i == j) table[i][j] = 0;
        else {
            table[i][j] = INT_MAX;
            for (int k = i; k < j; k++)
                table[i][j] = min(table[i][j],
                                    cost(i,k)+cost(k+1,j) + r[i]*c[k]*c[j]);
        }
    }
    return table[i][j];
}
```
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There are \(O(n^2)\) different subproblems (i.e. \((i, j)\) pairs).
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**Running time:** $O(n^3)$.
General Recipe

Formulate a mathematical *recurrence* that solves the problem.

**Ingredients:**

- “Base cases” that are trivial to solve.
- A clear way to break larger problems into “smaller” subproblems plus some easy to compute part.

Sort of a “computation by induction”. We can easily compute the solution if we know the solutions to smaller subproblems.
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**Running Time Analysis Template:**

\[(\text{# of possible subproblems}) \times (\text{time in one recursive call})\]

*Sometimes:* even faster, this is an upper bound.
Bottom-Up

The approach just shown is **Top-Down**: check a table before completing the calculation to see if it is already done.
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**Bottom-Up**: solve subproblems from “smallest” to “largest”.

```c
for (int len = 1; len <= n; len++)
    for (int i = 0, j = len - 1; j < n; i++, j++) {
        if (i == j) cost[i][j] = 0;
        else {
            cost[i][j] = INT_MAX;
            for (int k = i; k < j; k++)
                cost[i][j] = min(cost[i][j],
                                cost[i][k] + cost[k+1][j] + r[i]*c[k]*c[j]);
        }
    }
```

The loop ordering ensures when cost[i][j] is computed, the relevant cost[i][k] and cost[k+1][j] are already computed.
Comparison

Top-Down Advantage:
• Never have to think about how to cleverly build the table from the bottom up. Just memoize the natural recurrence.

Bottom-Up Advantage:
• Runs a bit faster. Not asymptotically faster, though.
• No concern about filling the call stack by recursing too deep.
• Can sometimes use a smaller table:
  example: if the recurrence has $f[i][j]$ only depending on $f[i-1][k]$ entries then we might only need to keep the single vector $f[i-1]$ when computing vector $f[i]$.

In the vast majority of contest problems, the top-down approach works fine.
Recovering a Solution

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To recover the sequence, just trace through the DP table.
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To recover the sequence, just trace through the DP table.

The outermost multiplication in the cheapest way to compute $A_i \cdot \ldots \cdot A_j$ is at the index $k$ achieving the min-value in the recurrence.

For this $k$, we compute $(A_i \cdot \ldots \cdot A_k) \cdot (A_{k+1} \cdot \ldots \cdot A_j)$. 
More Examples
Longest path in a DAG

Let $G$ be a DAG, a Directed Acyclie Graph with vertices $V$ and edges $E$. Let $s$ be a vertex.

Find the longest path (number of edges) starting at $s$. 
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Let $f(v)$ be the length of the longest path starting from $v$ and $N(v)$ all nodes $u$ such that $vu$ is an edge.
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$$f(v) = \begin{cases} 
0 & \text{if } N(v) = \emptyset \\
1 + \max_{u \in N(v)} f(u) & \text{otherwise}
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**Running Time:** $O(|V|^2)$: $|V|$ vertices, each with $\leq |V|$ neighbours.

**Maybe Better Analysis:** $O(|V| + |E|)$ if $N(v)$ is stored as an array.
Longest Common Subsequence

Let \( s = s_0 s_1 \ldots s_{n-1} \) and \( t = t_0 t_1 \ldots t_{m-1} \) be two sequences.

Find a longest common subsequence (LCS) between \( s \) and \( t \). For example, a LCS of \( s = \text{human} \) and \( t = \text{chimpanzee} \) is \( \text{hman} \).
Longest Common Subsequence

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Let $f(i, j)$ be the length of the LCS of $s_0 \ldots s_{i-1}$ and $t_0 \ldots t_{j-1}$.

$$f(i, j) = \begin{cases} 
0 & \text{if } i = 0 \text{ or } j = 0 \\
1 + f(i - 1, j - 1) & \text{if } s_i = t_j \\
\max(f(i - 1, j), f(i, j - 1)) & \text{otherwise}
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Running Time: \( O(nm) \)
Subset Sum

Given integers $a_1, \ldots, a_n \geq 0$ and some integer $\alpha \geq 0$, is there some $S \subseteq \{1, \ldots, n\}$ with $\sum_{i \in S} a_i = \alpha$?
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Let $f(k, b)$ be a boolean value that is true if and only if there is some $A \subseteq \{1, \ldots, k\}$ with $\sum_{i=1}^{k} a_i = b$. 

Running Time: $\mathcal{O}(n \cdot \alpha)$
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$$f(k, b) = \begin{cases} 
\text{TRUE} & \text{if } k = 0, b = 0 \\
\text{FALSE} & \text{if } k = 0, b \geq 1 \\
f(k - 1, b) & \text{if } k \geq 1, b \leq a_k - 1 \\
f(k - 1, b) \text{ OR } f(k - 1, b - a_k) & \text{if } k \geq 1, b \geq a_k 
\end{cases}$$

Running Time: $O(n \cdot \alpha)$
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**Running Time:** $O(n \cdot \alpha)$
Example Problem: Kattis - uxuhulvoting

https://open.kattis.com/problems/uxuhulvoting
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Basic idea: index the priests $i = 0 \ldots m - 1$ from youngest to oldest.
- For a priest $i$ and a “configuration” of stones $c$, let $f(i, c)$ be the final configuration of if priest $i$ plays optimally when presented with configuration $c$.
- As a base case, let $f(m, c) = c$ (here $m$ signals “no more priests”).
- Let $\alpha(c)$ be the set of 3 possible configurations we can get from $c$ by flipping one stone.
- For $0 \leq i < m$ we have $f(i, c)$ is the best of $f(i + 1, c')$ over all $c' \in \alpha(c)$.

Implementation detail: let $c$ be an integer between 0 and 7 (the binary string representing set of flipped stones). The bitwise XOR of $c$ with 1, 2, and 4 gives the three values in $\alpha(c)$. 
Example Problem: Kattis - pebblesolitaire2

https://open.kattis.com/problems/pebblesolitaire2
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https://open.kattis.com/problems/pebblesolitaire2

Basic idea:

• For a subset of positions $S$, let $f(S)$ be the minimum possible number that can be left if positions $S$ have pebbles.

• Recurrence
  ○ if no moves from $S$ the answer is $|S|$
  ○ otherwise, $f(S)$ is the minimum of $f(S')$ over all $S'$ that can be obtained by one move from $S$

• For $n$ holes, $2^n$ states and $O(n)$ moves/state so this takes $O(n \cdot 2^n)$ time.

• Don’t reset the memo table between inputs to save time.

Implementation detail: Represent $S$ by a single integer whose bits indicate the members of $S$. 
Example Problem: Kattis - race

https://open.kattis.com/problems/race
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https://open.kattis.com/problems/race

Basic idea:

- Similar to before, for a set of points $S$ and some point $v \in S$, let $f(S, v)$ be the minimum time to complete $S$ with $v$ being the last thing completed in $S$. This is $\infty$ if it is impossible.

- In the recurrence, compute $f(S, v)$ by guessing what task $u$ immediately preceded $v$.

- Quite a few simple low-level details to work out.

- $O(n2^n)$ subproblems, each with $O(n)$ recursive calls: $O(n^2 \cdot 2^n)$ time in total.