Bipartite Graphs and Partially Ordered Sets

Zachary Friggstad

Programming Club Meeting

Bipartite Recognition

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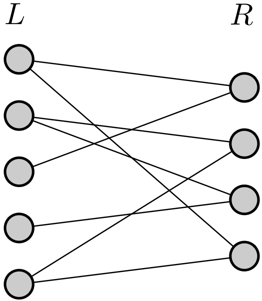

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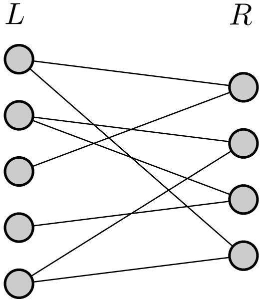


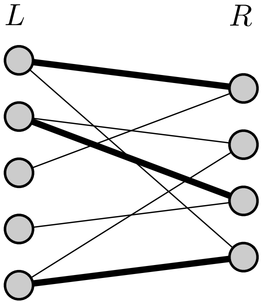
How to recognize bipartite graphs? Pick a vertex v, add v to L, add its neighbours to R, add their unprocessed neighbours to L, etc.

Now check if any edge has both endpoints on the same side.

```
typedef vector<vector<int>> graph;

graph g; //assume is already filled
vector<int> mark(g.size(), -1); //-1 means not processed

// uses a depth-first search
bool partition(int v, int side) {
   if (left[v] != -1) return mark[v] == side;
   mark[v] = side;
   for (auto u : g[v])
       if (!partition(u, 1-side)) return false;
   return true;
}

bool bip = true;
// run the dfs in each component
for (int u = 0; u < n && bip; ++u)
   bip &= (mark[u] != -1 || partition(u, 0));
//if bip == true, then {u : mark[u] = 0} is the left side
```

Bipartite Matchings

Let $G = (V, E)$ be a graph. A matching is a subset $M \subseteq E$ such that each vertex $v \in V$ is the endpoint of *at most* one edge in M.



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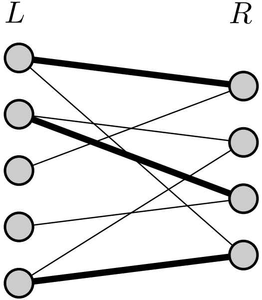

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Optimization Question

Find the largest possible matching M.

Can find in polynomial time for **any** graph, but the algorithm is a bit too intricate for the contest setting.

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Basic Approach
Let $n = |V|, m = |E|$.

There is a $O(n + m)$-time algorithm that does the following:

Given a matching M, either finds a larger matching or else correctly determines M is a maximum-size matching.

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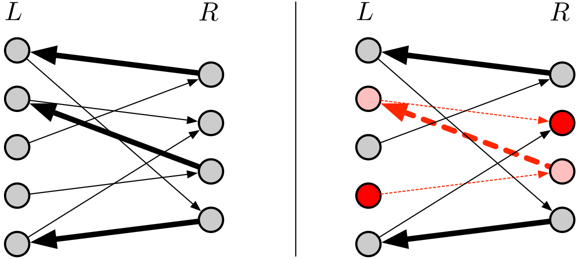
Iterating the procedure starting with $M = \emptyset$ finds a maximum matching in $O(mn + n^2)$ time.

Augmenting Paths

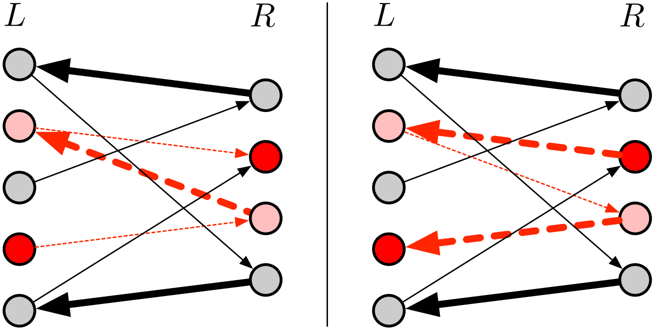
Given a matching M, direct all edges e as follows:

- From L to R if $e \notin M$
- From R to L if $e \in M$



Find an M-alternating path P: a path from an unmatched vertex in L to an unmatched vertex in R.

Flip the directions of the edges in P: the matching is larger.



Lemma

If M is not a maximum matching, there is an M-alternating path.

Idea:

Let M^* be any matching. Then $M \cup M^*$ (keep doubles) is comprised of paths and cycles that alternate between M and M^*.

If $|M| < |M^*|$, then some path starts and ends with an edge in M^*. This is an M-alternating path.

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Pseudocode:

- $M = \emptyset$
- While there is an M-alternating path P
- Update $M \leftarrow M \oplus P$ (toggle/flip edges of P)
- Return M

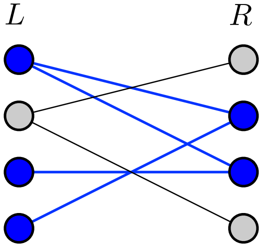
```
int left; //# nodes on left
graph g;  //say L is indexed from 0 to left-1
vector<int> match(g.size(), -1), seen(left, -1);

bool augment(int u, int cno) { //find a path via dfs
  if (seen[u] == cno) return false;
  seen[u] = cno;
  for (auto v : g[u])
    if (match[v] == -1 || augment(match[v], cno)) {
      match[v] = u; match[u] = v; // flip the edges
      return true;
    }
  return false;
}

int match() {
  int cnt = 0;
  // can show we only need to search from each vertex once
  for (int u = 0; u < left; ++u)
     if (augment(u, u)) ++cnt;
  return cnt;
}
```

Hall's Theorem.

There is a matching M of size $|L|$ if and only if for every $X \subseteq L$, the number of nodes in R adjacent to some vertex in X is at least $|X|$.



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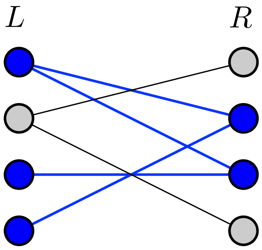
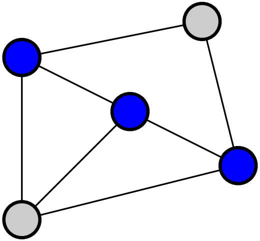


Picture: No way to match all three highlighted nodes in L.

To find such an X, if max matching $|M| < |L|$ let X be all nodes on L reachable from M-alternating paths from unmatched nodes.

Exercise: Prove such X has $< |X|$ neighbours.

A vertex cover is a set C of nodes so each edge e has at least one endpoint in C.



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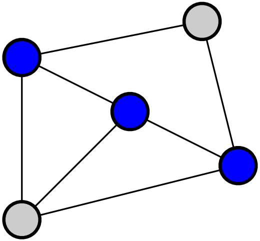

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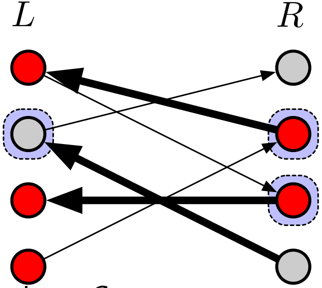
Theorem (Vizing)

If C is a min vertex cover and M a max matching in a bipartite graph, then $|C| = |M|$.

Again, let M be a max-matching.

Let S be the set of all vertices reachable by an M-alternating path starting at the unmatched nodes on the left.

Let $C = (L - S) \cup (R \cap S)$. This is a vertex cover.



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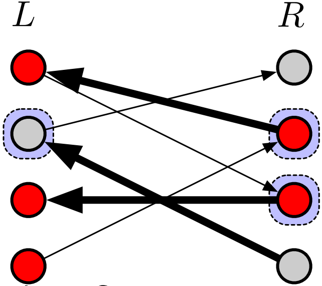

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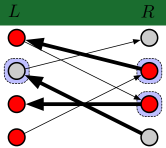
Side Note: Can show $L \cap S$ is a "Hall Set" (i.e. fewer than $|L \cap S|$ neighbours) if $|M| < |L|$.

Steps to the proof.



Showing C is a vertex cover

Show no edge $e = uv$ has $u \in L \cap S, v \in R - S$ using definition of "reachability" for S (two cases, if $e \in M$ or $e \notin M$).

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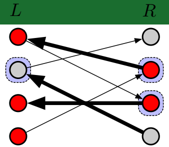

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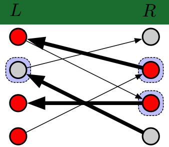

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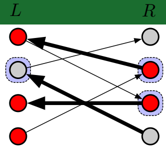

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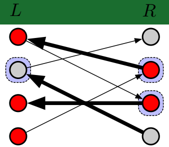

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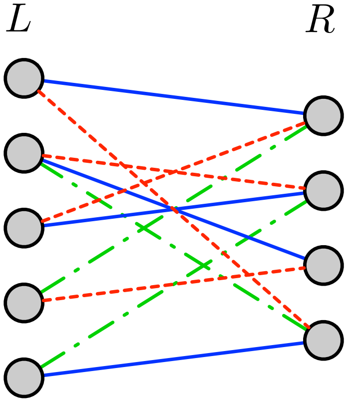
Therefore $|C| = |M|$. But any vertex cover needs $\geq |M|$ vertices just to cover M, **so C is a minimum vertex cover**.

Edge Colouring

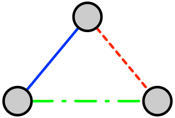
Problem: Colour each edge of a graph so every colour is a matching.



Degree Bounds

Obvious Bound: If Δ is the maximum "degree" of a vertex (# of incident edges), then we need $\geq \Delta$ colours.

Sometimes more, here $\Delta = 2$ but 3 colours are needed:



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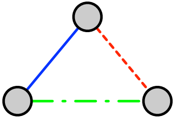

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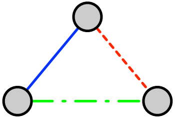

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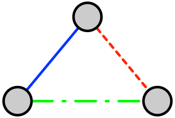

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König: In a bipartite graph, Δ colours suffice.

Algorithm
Colour the edges one at a time.

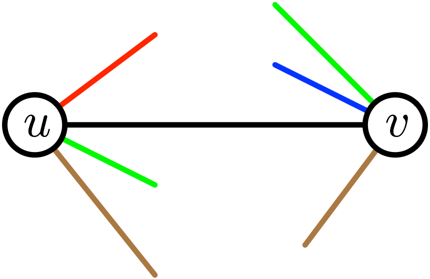
When processing $e = uv$, if there is an available colour, use it!

- That is, if some colour has neither u nor v matched by that colour so far, then use that colour for e.

Otherwise, we will modify the current colouring in $O(n)$ time to ensure there is an available colour.

Modifying the Colouring

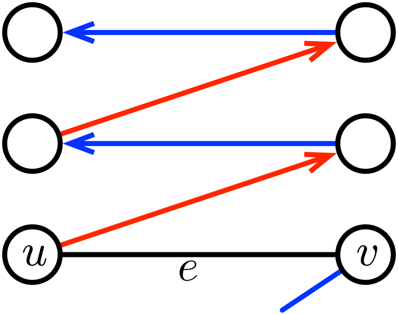
Say we are trying to colour $e = uv$ but none of the Δ colours are free on both u and v.



We still know some colour, say *blue*, is not used on u and some colour, say *red*, is not used on v.

Modifying the Colouring

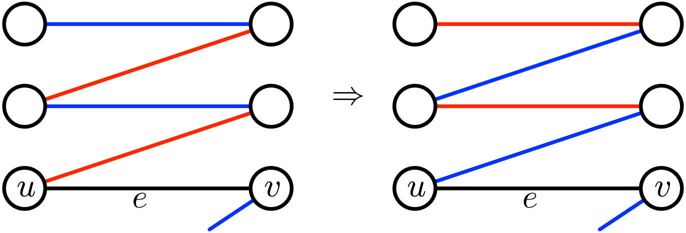
Consider the *maximal* path from u that alternates between *blue* and *red* edges.



This path does not contain v since it goes $L \to R$ along *red* edges.

Modifying the Colouring

Swap the colours of the *blue* and *red* on the path from u.



Now we can colour e *red*.

Running Time: $O(m \cdot n)$.
The colour "swapping" along the path takes $O(n)$ time. Done at most once per edge added.

Partially Ordered Sets

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- **Reflexivity**: For $a \in X$, $a \preceq a$.
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Write $a \prec b$ for $(a \preceq b$ and $a \neq b)$.

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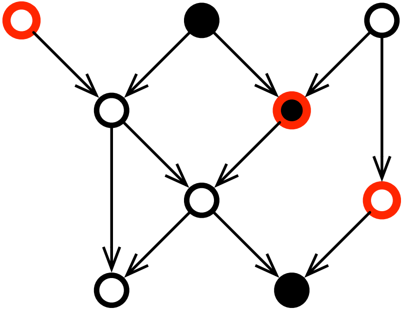
3. Let $G = (X; E)$ be a directed, acyclic graph. Say $u \preceq v$ if there is a $u - v$ path.
4. Let $G = (X; E)$ be a graph with nonzero edge distances. Fix $r \in V$. Say $u \preceq v$ if some shortest $r - v$ path passes through u.

Can view as a directed, acyclic graph. We can omit missing edges inferred by transitivity (a Hasse diagram).



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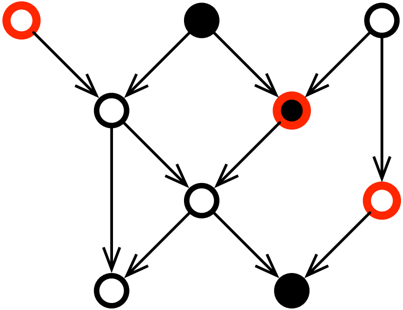

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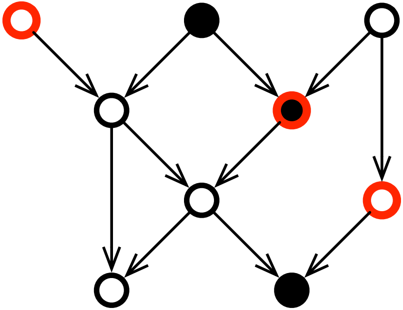

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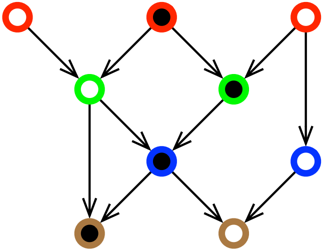
An antichain is a set $A \subseteq X$ where no two $a, b \in X$ have $a \prec b$ **or** $b \prec a$. i.e. no two items in A are comparable. (e.g. thick red outline)

Theorem

Longest chain = minimum # of antichains to cover all nodes.



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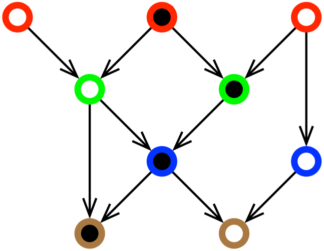

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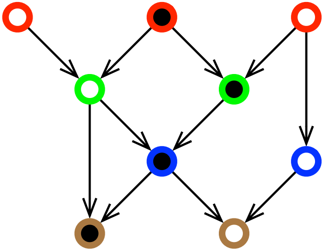

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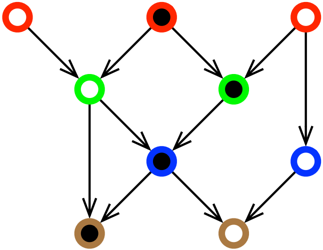

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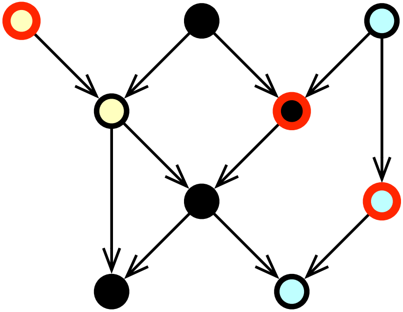
Then for all $k \in \mathbb{Z}$, $\{v : \ell[v] = k\}$ is an antichain.

Can compute in $O(|V| + |E|)$ time.

Theorem (Dilworth's Theorem)

Largest antichain = minimum # of chains to cover all nodes.

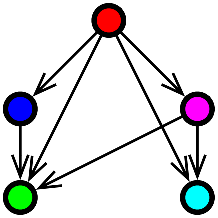


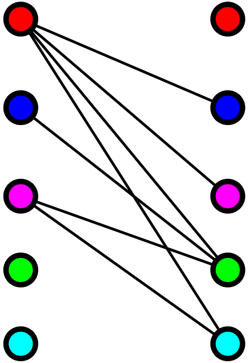
Figure

Nodes with a red outline form a maximum antichain. The colours filling the nodes partition the nodes into three chains.

Form an auxiliary bipartite graph: a copy of X on each side.

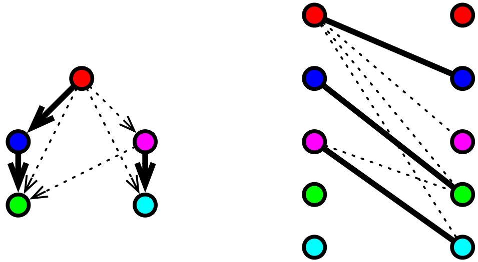
Add edge $u \in L$ to $v \in R$ if $u \prec v$ (but **NOT** $u = v$).





If starting with a Hasse diagram, don't forget edges implied by transitivity!

Bipartite matching of size $k \equiv$ chain cover of size $n - k$.



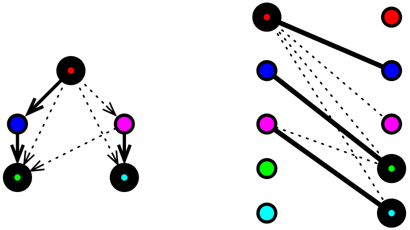
Note a singleton node $v \in X$ forms its own chain: this corresponds to no copy of v being matched.

So find a maximum bipartite matching to find minimum # of chains to cover all nodes.

Running time: $O(n \cdot m)$ where m is the number of $u \prec v$ pairs.

Finally, Maximum Antichains

In the bipartite graph, let C be a min. vertex cover (thick vertices).



Can show minimality of C and transitivity means no $v \in X$ has both copies in C.

Back in the poset, C covers all directed edges and has size $|M|$.

Therefore, $X - C$ (middle nodes in the poset above) is an antichain of size $n - |M| = (\#$ of chains in cover).

UVa 11396 - Claw Decomposition

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So just check if G is bipartite!

UVa 11331 - Joys of Farming

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Dynamic programming!

Built this table:
$f[i,b] = \texttt{true}$ if it is possible to assign precise b bulls to the first i components, `false` otherwise.

Can fill in $O((b+c)^2)$ time.

UVa 10122 - Mysterious Mountain

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A form of Bipartite matching?

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Binary search!

Running Time: $O(N^3)$ time to build the graph, $O(N^3 \cdot \log N)$ to binary search (just search over the $O(N^2)$ different edge values).

Unnecessary Exercise: Add edges in increasing order until there is a matching. A careful approach can process each edge in $O(N)$ time.

UVa 10615 - Rooks

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Let $G = (L \cup R; E)$ be a bipartite graph where:

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Compute a minimum edge colouring in $O(N^3)$ time.

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See also UVa 12668 for a related problem.

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Solution: Compute the longest nonincreasing sequence of heights in this sequence in $O(m \log m)$ time!