Programming Club Notes

Least Common Ancestor Queries

Zachary Friggstad

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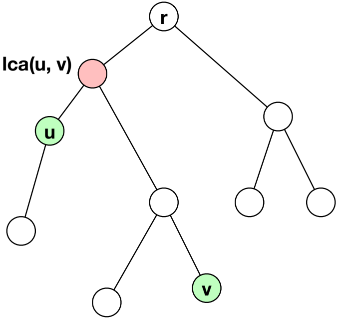
Basic problem: given a tree $T = (V; E)$ rooted at a vertex r and two vertices u, v, find the least common ancestor of u and v.

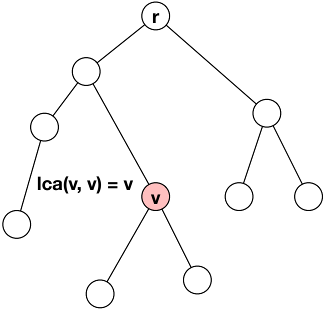


This is the deepest vertex that lies somewhere above (i.e. is an ancestor) of both u and v.

Denoted $\textbf{lca}(u, v)$

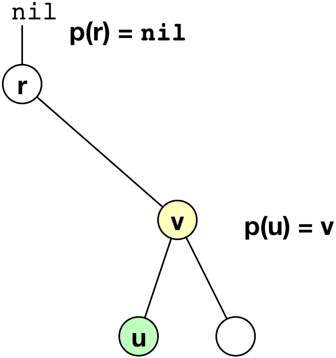
Observe $\textbf{lca}(v, v) = v$.

Note for u, v that $\textbf{lca}(u, v) = u$ if and only if u lies on the $r - v$ path in the tree T.

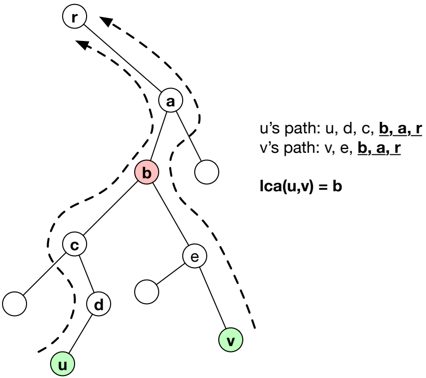


Also, $\textbf{lca}(r, v) = r$ for any v (recall $r ==$ root).

The only thing we assume is that we know the parent $p(v)$ of any vertex v in the tree. Say $p(r) = \texttt{nil}$.



A simple algorithm: construct the $v - r$ path by crawling up the tree (i.e. following $p()$). Follow the $u - r$ path by crawling up the tree. The start of the common suffix of these paths is $\mathbf{lca}(u, v)$.



A single query takes $O(n)$ time.

A Faster Approach

If we anticipate many queries, we can do some preprocessing to speed them up.

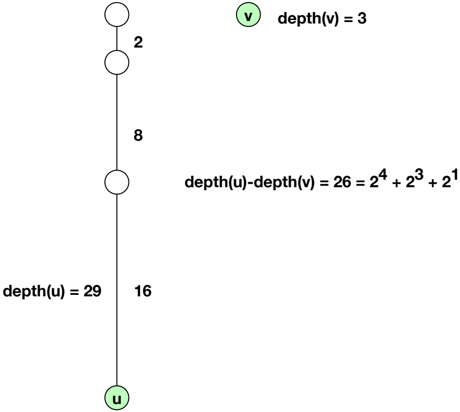
Preprocessing: compute the following

- For each $v \in V$
  $\mathbf{depth}(v)$: the distance to the root, so $\mathbf{depth}(r) = 0$
- For each $v \in V$ and each $0 \leq i \leq \log_2 n$
  $p(v, i)$: the vertex that is 2^i steps above v.
  If $\mathbf{depth}(v) < 2^i$, then $p(v, i) = \texttt{nil}$

We can compute all of these values in $O(n \log n)$ time (see this later).

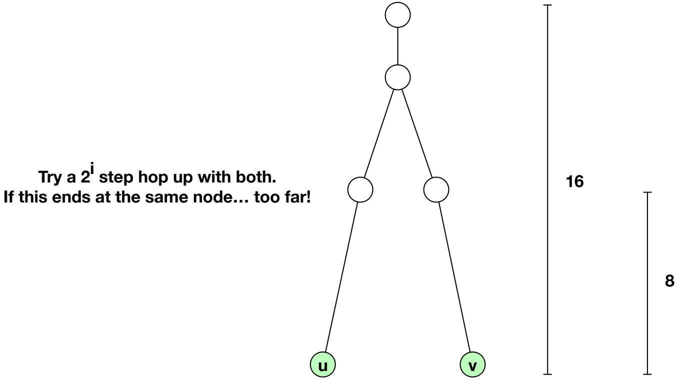
Given these values, here is how to compute **lca**(u, v).

Rough idea: walk the deepest node upward by taking 2^i-length steps for decreasing i but only if this does not overshoot **depth**(v).



After, u, v have the same depth. If $u = v$ now, we are done!

Otherwise, walk u, v upward simultaneously taking 2^i-length steps for decreasing i but only if this does not merge the nodes (don't want to merge too far up the tree).



Once done, **lca**(u, v) is the common parent of u and v.

Step 1: Walk the deeper node up the depth of the other.

- Assume $\textbf{depth}(u) \geq \textbf{depth}(v)$ (swap u, v otherwise).
- **for** $i = \log_2 n$ down to 0
  - **if** $\textbf{depth}(u) - 2^i \geq \textbf{depth}(v)$ **then** $u \leftarrow p(u, i)$.

Now u, v are at the same depth and they have the same **lca** as the old u, v.

Step 2: Walk u, v up simultaneously until they have a common parent.

- **if** $u = v$ already, **then** return v
- **for** $i = \log_2 n$ down to 0
  - **if** $p(u, i) \neq p(v, i)$ **then** $u, v \leftarrow p(u, i), p(v, i)$
- **return** $p(u, 0)$ (i.e. the parent of u)

Note the **if** in the loop also catches the case when $p(u, i) = p(v, i) = \texttt{nil}$ (i.e. $\textbf{depth}(u) < 2^i$).

Entire query is processed in $O(\log n)$ time!

Using just $p(u)$ information, we can compute $\textbf{depth}(u)$ and $p(u,i)$ entries with dynamic programming!

$$\textbf{depth}(u) = \begin{cases} \textbf{depth}(p(u)) + 1 & \text{if } u \neq r \\ 0 & \text{if } u = r \end{cases}$$

For $p(u,i)$, apart from obvious base cases just take two 2^{i-1} hops up.

$$p(u,i) = \begin{cases} p(u) & \text{if } i = 0 \\ \texttt{nil} & \text{if } p(u,i-1) = \texttt{nil} \\ p(p(u,i-1), i-1) & \text{otherwise} \end{cases}$$

If you are truly laze, you don't even need to fill these values by preprocessing, just fill the table on the fly! Total time spent filling all DP table entries is $O(n \log n)$.

Practical Tips

Usually vertices in a graph are numbered 0 through $n-1$. Use -2 to represent `nil` and -1 to represent *not yet computed* in the DP table.

```
#define MAXN 100000
#define LOGN 17 // the smallest k such that 2^k > MAXN

int depth[MAXN];
int p[MAXN][LOGN];

// initialize DP tables to -1: entries not yet computes
memset(depth, -1, sizeof(depth));
memset(p, -1, sizeof(p));

// read in the parent pointers into p[v][0] entries,
// remembering to initialize p[root][0] = -2, i.e. nill

// start all "i  from log n down to ..." loops at LOGN-1
```