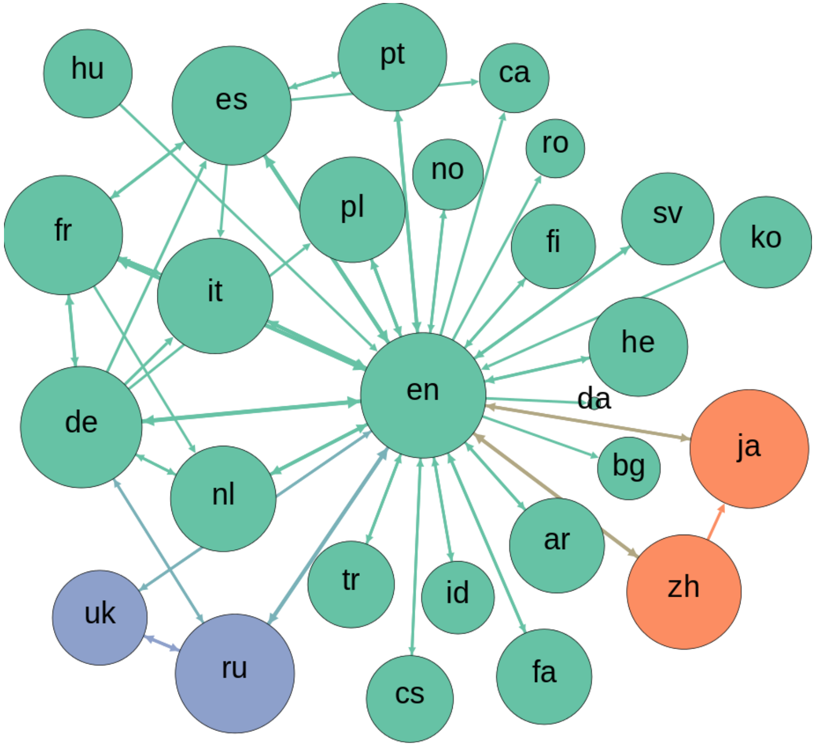
Graph Theory Crash Course II

2015

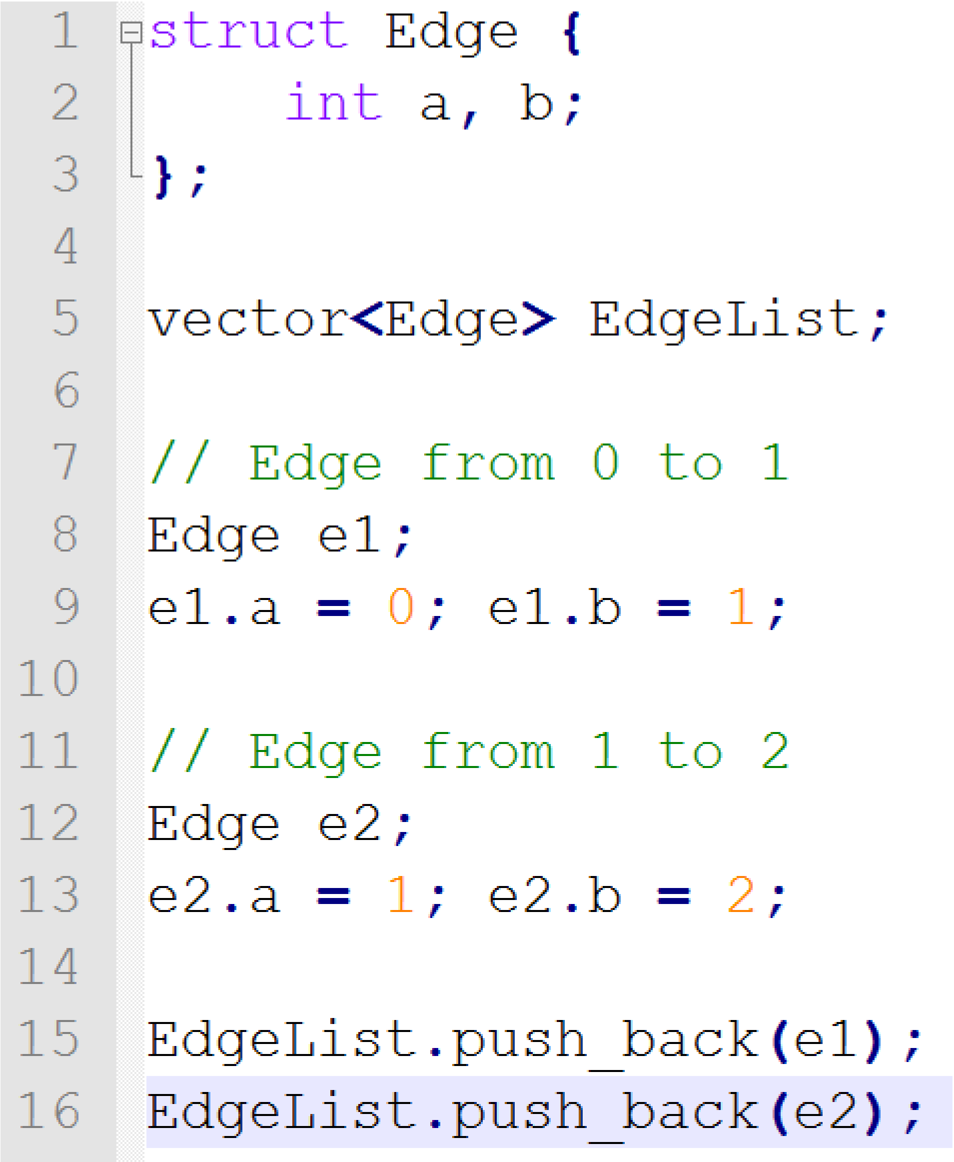
Graph Representation Review

- Edge List
- Adjacency Matrix
- Adjacency List



Edge list

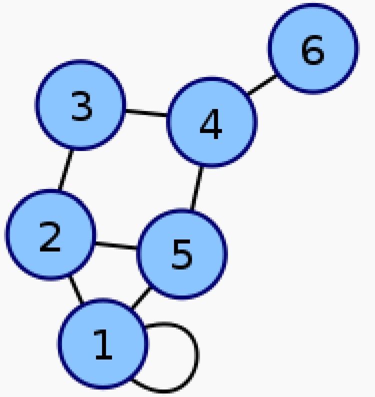
- Simply make a list (or vector) of pairwise relations.



```
struct Edge {
    int a, b;
};

vector<Edge> EdgeList;

// Edge from 0 to 1
Edge e1;
e1.a = 0; e1.b = 1;

// Edge from 1 to 2
Edge e2;
e2.a = 1; e2.b = 2;

EdgeList.push_back(e1);
EdgeList.push_back(e2);
```

Adjacency Matrix

- Make a table. Rows correspond to the source node, columns to the destination node.
- A 1 in row R and column C means that the edge R->C exists.

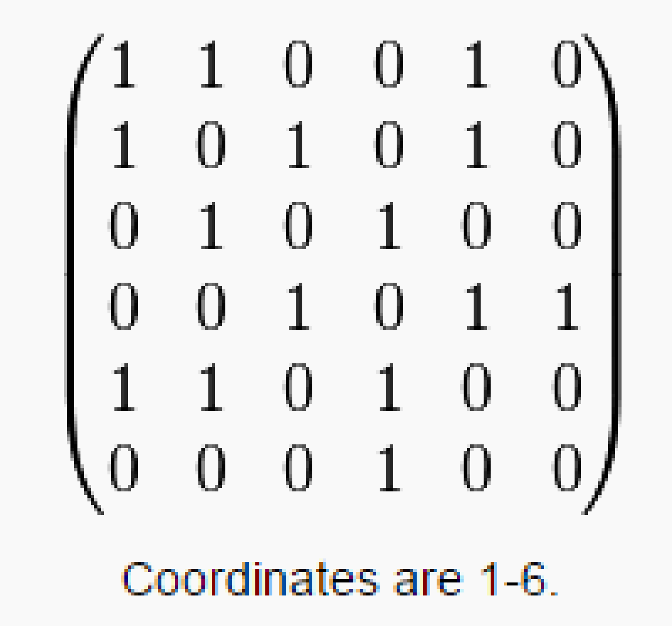


$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Coordinates are 1-6.

Adjacency Matrix

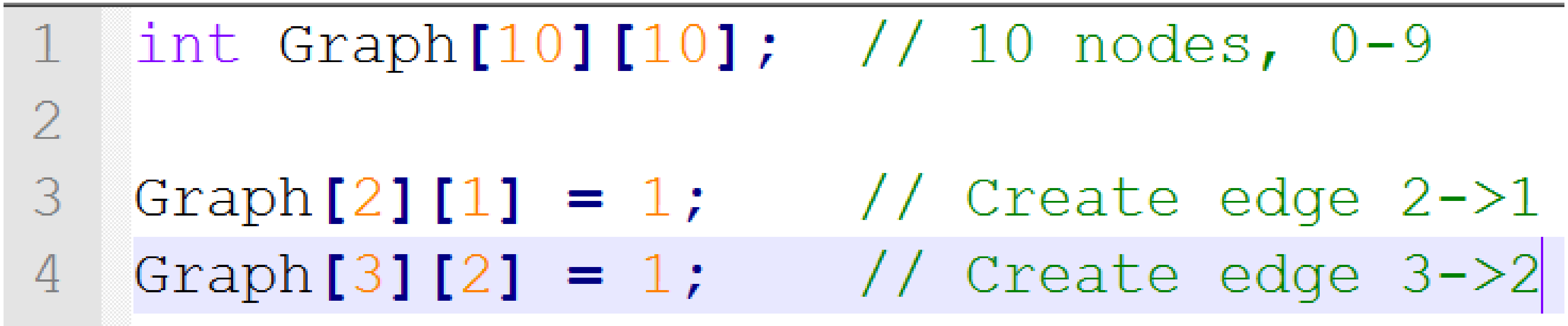
- Undirected graph: if a->b exists, so does b->a
  - Therefore, matrix symmetric.
- Weighted graph
  - May replace 1 with the edge weight.



$$\begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Coordinates are 1-6.

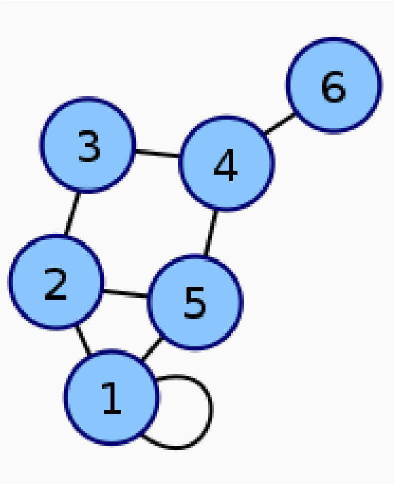
Adjacency Matrix



```
int Graph[10][10];   // 10 nodes, 0-9

Graph[2][1] = 1;     // Create edge 2->1
Graph[3][2] = 1;     // Create edge 3->2
```

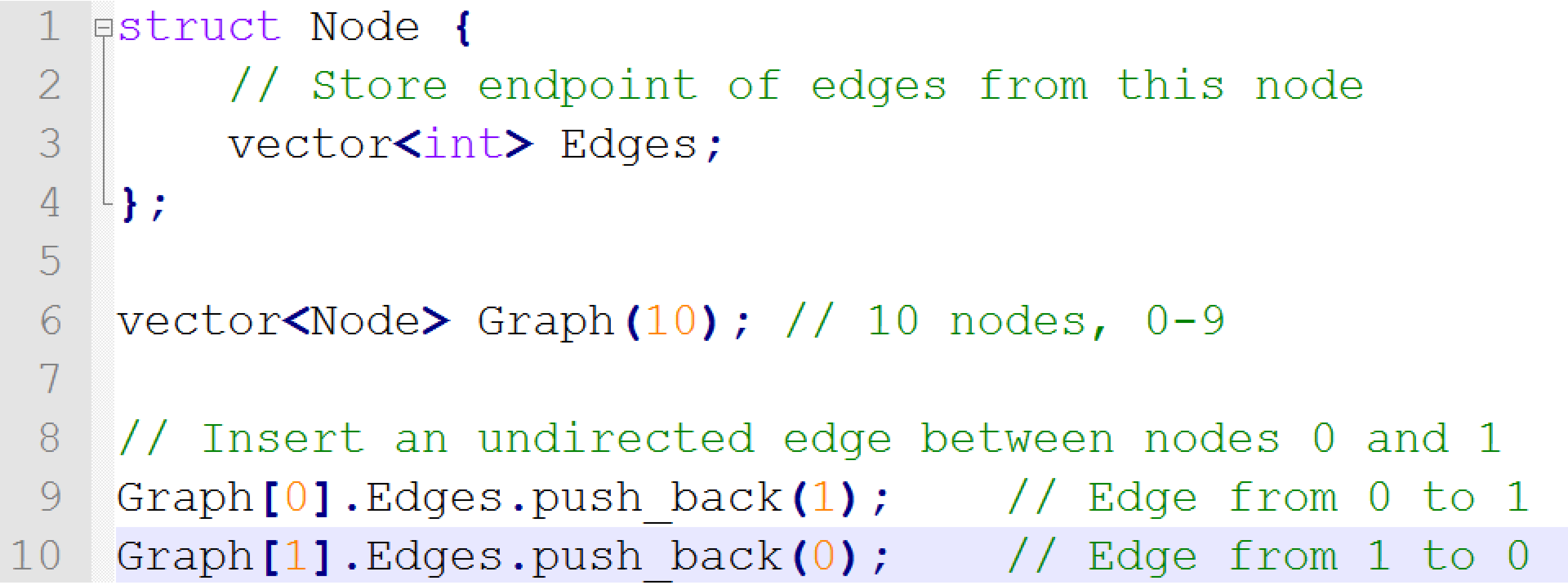
Adjacency Lists

- Each node stores the edges that extend from that node.
- Example:
  - Node 1 stores:
    - Edge to 5
    - Edge to 1
    - Edge to 2



- For undirected graphs, we need to be careful to add the reverse edges too.

Adjacency Lists

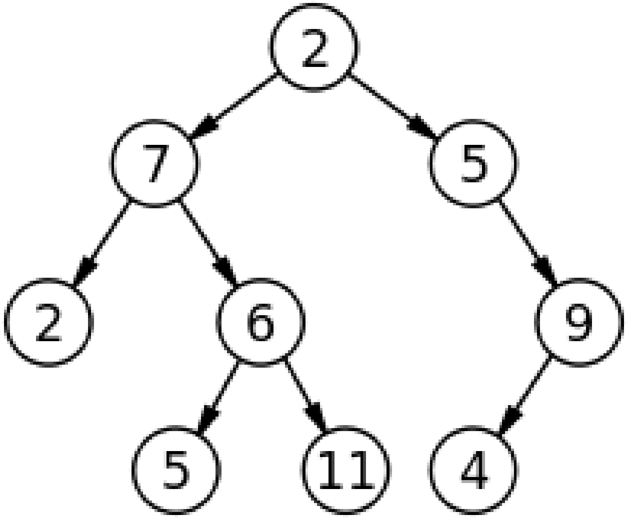


```
struct Node {
    // Store endpoint of edges from this node
    vector<int> Edges;
};

vector<Node> Graph(10); // 10 nodes, 0-9

// Insert an undirected edge between nodes 0 and 1
Graph[0].Edges.push_back(1);    // Edge from 0 to 1
Graph[1].Edges.push_back(0);    // Edge from 1 to 0
```

Which to Use?

- Depends on the algorithm
  - Some algorithms are more naturally implemented on a particular representation.
  - Some queries are inefficient on edge lists, e.g. is there an edge between two given nodes?
- Depends on the graph
  - Adjacency matrix inefficient for sparse graphs, always takes O(V^2) space.

Graph Algorithms



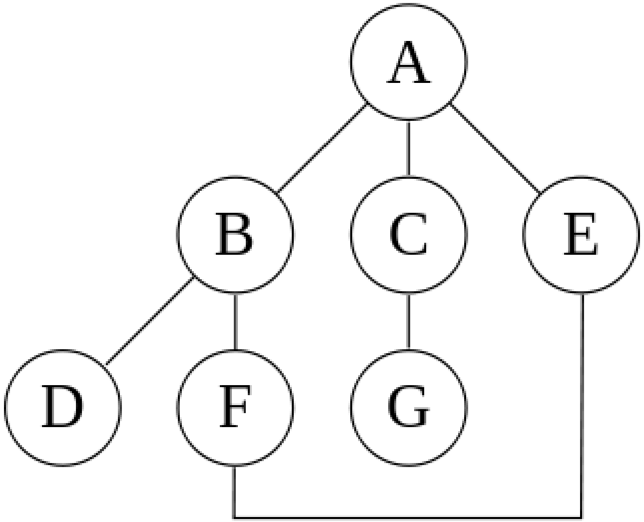
- Depth-first search (DFS)
- Breadth-first search (BFS)
- DFS & BFS Variants
- Minimum spanning tree (MST)
  - Kruskal's algorithm & Prim's algorithm
- Single-source shortest path (SSSP)
  - BFS, Djikstra's algorithm, Bellman-ford
- All-pairs shortest path (APSP)
  - Floyd-Warshall
- Bipartite Graph
  - Bipartite graph check (BFS), Maximum matching
- Flow algorithms

Graph Algorithms: DFS & BFS

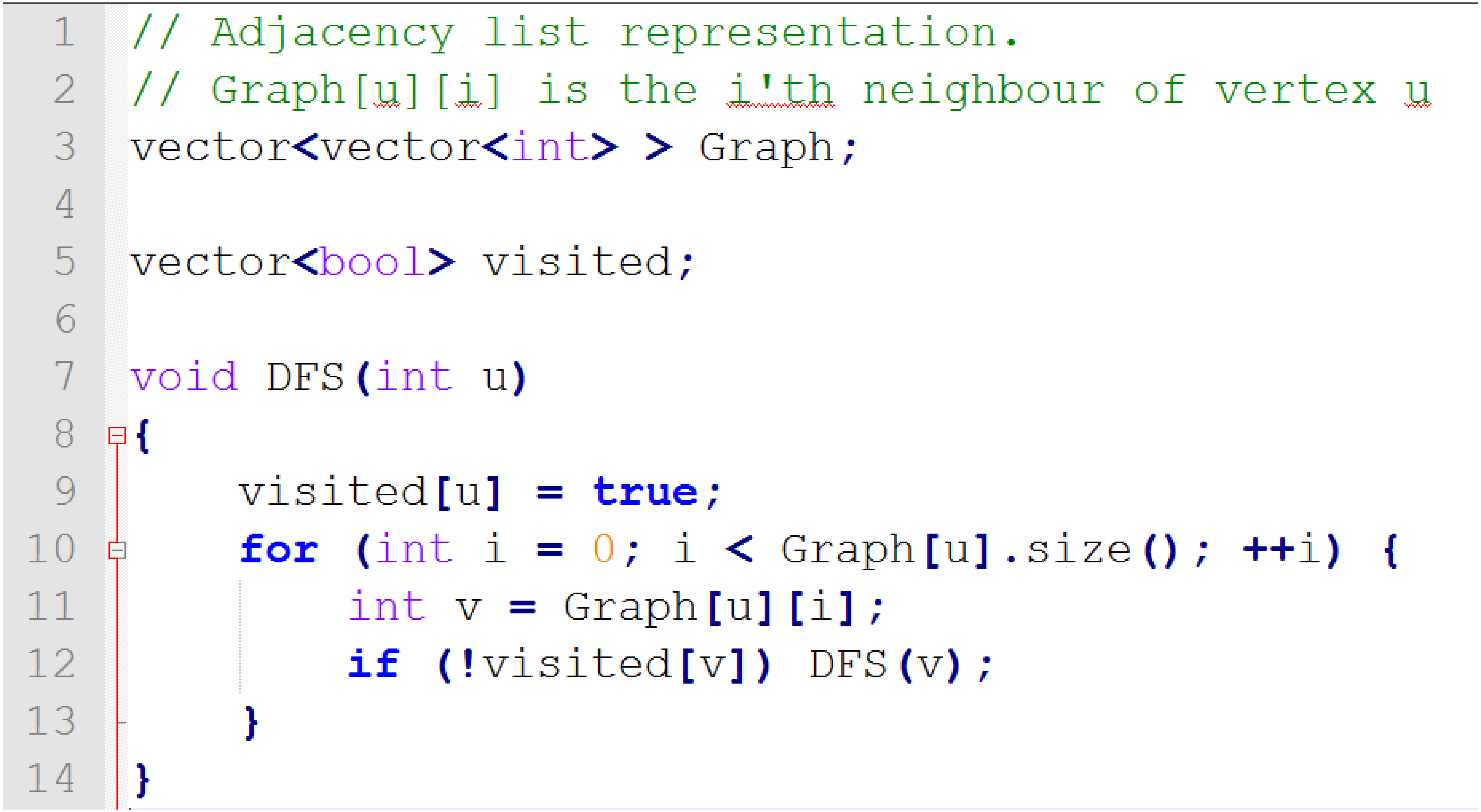
- Graph Search
- Single-source shortest path, unweighted (BFS)
- Strongly-connected components
  - Undirected: DFS / BFS
  - Directed: Tarjan’s algorithm
- Topological sort (DFS)
- Finding articulation points and bridges

Depth-First Search

- DFS on vertex U:
    - – Mark U as visited.
    - – For each neighbour V of U that has not been visited:
        - DFS on vertex V



Depth-First search



```
// Adjacency list representation.
// Graph[u][i] is the i'th neighbour of vertex u
vector<vector<int> > Graph;

vector<bool> visited;

void DFS(int u)
{
    visited[u] = true;
    for (int i = 0; i < Graph[u].size(); ++i) {
        int v = Graph[u][i];
        if (!visited[v]) DFS(v);
    }
}
```

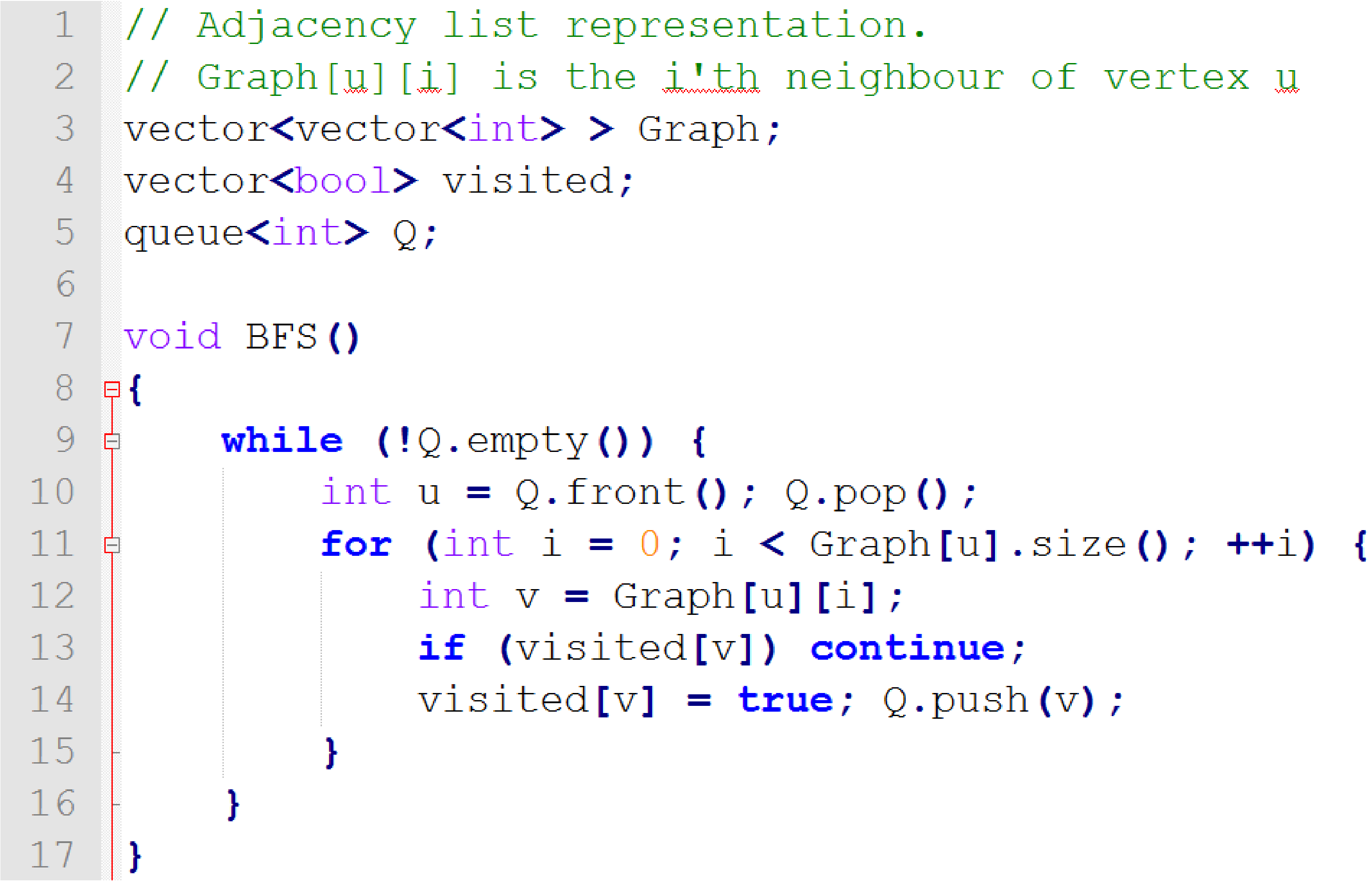
Breadth-First Search

- Use Queue to decide which node to visit next.
- BFS:

Loop:

- If Q empty, done!
- Get and remove vertex U at front of Q
- For each neighbour V of U:
  - If V has not been visited:
    - Set V to visited
    - Enqueue V
- Goto: Loop

Breadth-First Search

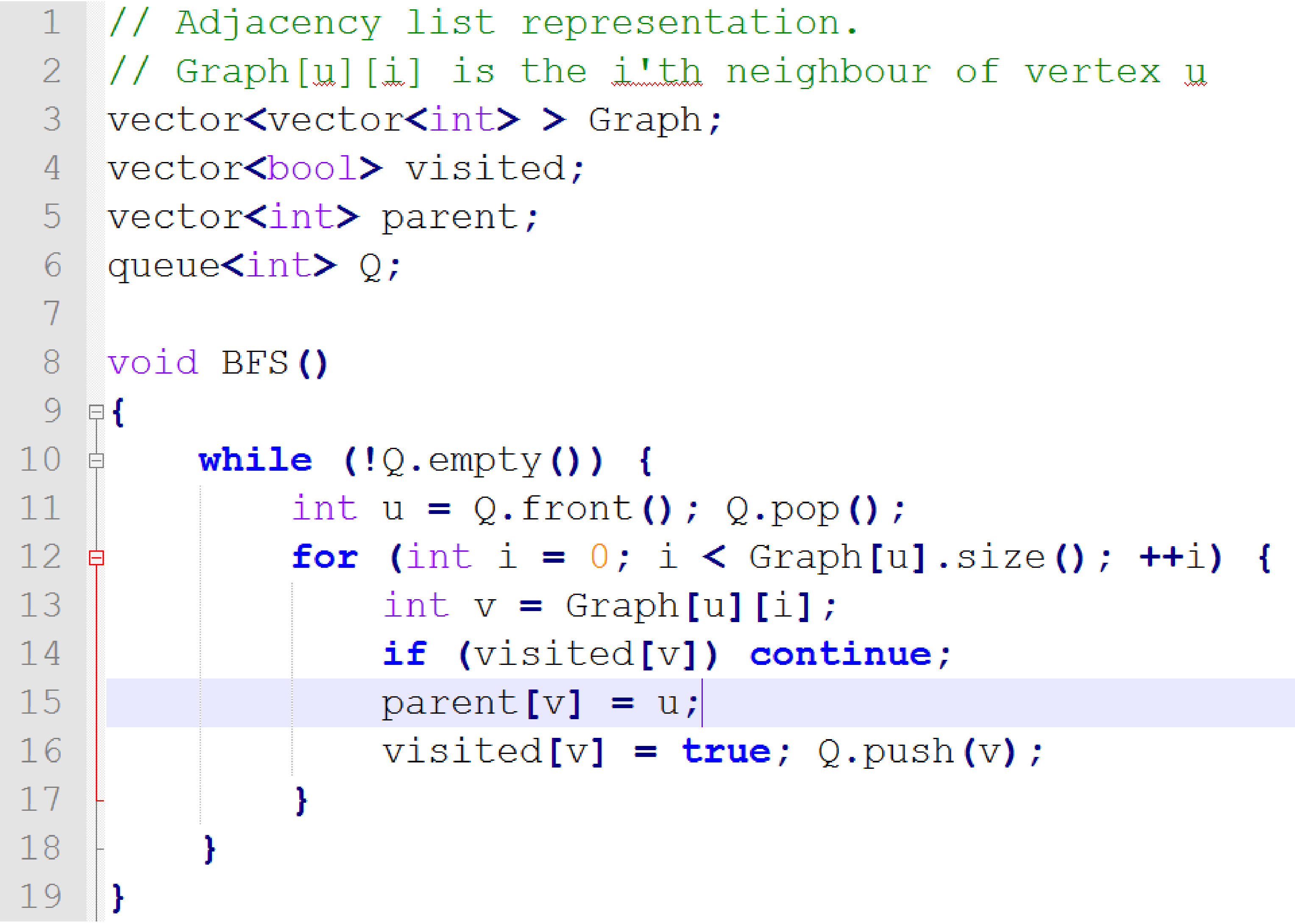


```
// Adjacency list representation.
// Graph[u][i] is the i'th neighbour of vertex u
vector<vector<int> > Graph;
vector<bool> visited;
queue<int> Q;

void BFS()
{
    while (!Q.empty()) {
        int u = Q.front(); Q.pop();
        for (int i = 0; i < Graph[u].size(); ++i) {
            int v = Graph[u][i];
            if (visited[v]) continue;
            visited[v] = true; Q.push(v);
        }
    }
}
```

SSSP with BFS

- Works for unweighted graph, or where edges all have weight 1.
- Keep around a vector of “parents”
- Whenever you visit a node, record the node you came from as the parent
- Follow the parents to find the shortest path

SSSP with BFS



```
// Adjacency list representation.
// Graph[u][i] is the i'th neighbour of vertex u
vector<vector<int> > Graph;
vector<bool> visited;
vector<int> parent;
queue<int> Q;

void BFS()
{
    while (!Q.empty()) {
        int u = Q.front(); Q.pop();
        for (int i = 0; i < Graph[u].size(); ++i) {
            int v = Graph[u][i];
            if (visited[v]) continue;
            parent[v] = u;
            visited[v] = true; Q.push(v);
        }
    }
}
```

Strongly-Connected Components

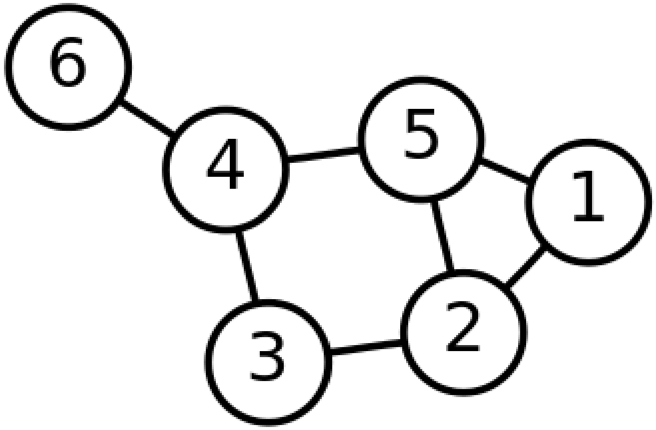
- SCC is a subset of nodes where there exists a path between any pair of nodes in the subset.
- For an undirected graph, can use DFS or BFS to find them.
- For a directed graph, use Tarjan’s algorithm (variant of DFS)

SCC with DFS

- All nodes we reach during a single run of DFS are in the same SCC
- Simply run DFS on an unvisited node. All nodes the DFS visits are members of the newly found SCC

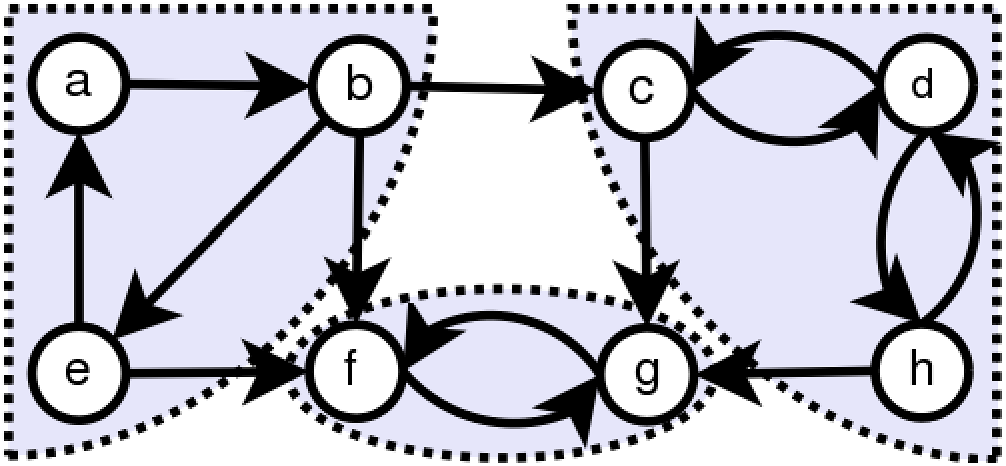
SCC with DFS

- For each node u:
  - If u has not been visited:
    - Report new SCC
    - DFS(u)



SCC with Tarjan's Algorithm

- Finds SCCs in directed graphs
- Variant of DFS, we'll get back to this one later.



Articulation points & Bridges

- Articulation point: Node that, if removed, disconnects the graph.
- Bridge: Edge that, if removed, disconnects the graph.
- Of strategic importance (cut off enemy supply lines, etc.)
- How to find these?

Articulation Points & Bridges

- Simple Method:
  - First, run DFS to verify graph connected.
  - For each node:
    - Remove the node.
    - Run DFS to see if the graph has been disconnected.
- O(V(V+E))

Articulation Points & Bridges

- More efficient method: Modified DFS
- Introduce two node labels: DFS_num and DFS_low
- DFS_num: Iteration on which we first saw this node.
- DFS_low: Smallest DFS_num we can reach in the DFS subtree below this node.