A Polynomial Time Algorithm for Strong Edge Coloring of Partial k-Trees

Mohammad R. Salavatipour *
Department of computer Science, University of Toronto
10 King's College Rd., Toronto, ON, M5S 3G4, Canada
mreza@cs.toronto.edu

Abstract

A matching M in a graph is called induced if there is no edge in the graph connecting two edges of M. The strong edge coloring problem is to find an edge coloring of a given graph with minimum number of colors such that each color class is an induced matching. This problem is known to be NP-complete, even in very restricted cases. Here, we show that it can be solved in polynomial time on graphs with bounded treewidth, i.e partial k-trees. This answers an open question of [14].

1 Introduction

For a given graph $G = (V, E)$, an *induced matching* is a set $M \subseteq E$ such that there is no edge in E connecting two edges of M. That is, the subgraph induced by the vertices of M is precisely M. A strong edge coloring of G is an assignment of colors to the edges of G such that every color class is an induced matching. Obviously, the goal is to use as few colors as possible. Perhaps the most important open question on the strong edge coloring problem is the conjecture of Erdös and Nešetřil (see [7], [8]) , which states that every graph of maximum degree Δ has a strong edge coloring with at most $\frac{5}{4}\Delta^2$ colors. It is not difficult to see that strong edge coloring of a graph G is equivalent to coloring the square of the line graph of G, denoted by $L(G)^2$, where the square of a graph is obtained by adding edges between every two vertices in that graph of distance exactly two (see [6]). The problem of coloring the square of a graph has many applications, for example, in the Frequency Channel Assignment problem, or in approximating the Hessian matrices of certain nonlinear functions using a minimum number of gradient evaluations [15]. Coloring the square of a graph is a well studied problem and is proved to be NP-complete even for planar graphs of bounded degree.

Although there are efficient algorithms for the standard matching problem, finding a maximum induced matching (called MIM from now on) is NP-complete, even for bipartite graphs with maximum degree 4 [20] and for 3-regular graphs [13]. On the other hand, the MIM problem is solvable in polynomial time for trees [9], chordal graphs [6], circular arc graphs [10], trapezoid graphs, co-comparability graphs [11], and weakly chordal graphs [21]. The main idea of the most of these algorithms is to show that for a class $\mathcal{C}$ of graphs, if $G \in \mathcal{C}$ then $L(G)^2 \in \mathcal{C}$ as well. This immediately yields a polynomial time algorithm for the MIM problem, because an induced matching of G is equivalent to an independent set

*Supported by Research Assistantship, Department of computer science, University of Toronto.

of $L(G)^2$ and finding a maximum independent set is polynomially solvable for each of these classes of graphs.

A similar idea reduces the strong edge coloring problem to the standard vertex coloring problem, which is polynomially solvable for these graphs. Therefore, the strong edge coloring problem can be solved in polynomial time on chordal graphs and co-comparability graphs. This was noticed by Mahdian [14]. But the strong edge coloring problem is NP-complete in general, and remains NP-complete even for bipartite graphs with girth at least 4 [14].

Based on the results of [1] and [5], one can easily find a polynomial (and in fact linear) time algorithm for MIM for partial k-trees, since this problem can be expressed in Extended Monadic Second Order Logic (EMSOL). For the strong edge coloring problem the situation is more complicated. It is not difficult to express the strong edge coloring problem in EMSOL as long as the number of colors (and therefore the maximum degree of the graph) is bounded. But no one knows whether this problem can be expressed in EMSOL if the maximum degree of the graph is not bounded, and so we need a different approach for this problem in general case.

In this paper, we solve this problem by removing the restriction of having bounded maximum degree. In other words, we show that the strong edge coloring problem can be solved in polynomial time on graphs with bounded treewidth, i.e. partial k-trees. This answers one of the open questions of [14].

2 Preliminaries and Definitions

We consider only simple undirected graphs. Let M be an induced matching of a graph $G = (V, E)$. If $H \subseteq G$ is a subgraph of G, by $M|H$ we mean the set of edges of M that are in H. Clearly, $M|H$ is an induced matching of H as well.

A strong edge coloring of $G = (V, E)$ is an assignment of integers $f : E \longrightarrow \mathbb{N}$ such that for every two edges $e = (u, v)$ and $e' = (u', v')$, if either $(u, v') \in E$ or $(u', v) \in E$ or one of the end points of e and e' are common, then $f(e) \neq f(e')$. The minimum number of colors (integers) required in a strong edge coloring of G is called the *strong chromatic index* of G.

The notion of treewidth was introduced by Robertson and Seymour [16, 17, 18] in their long and deep series of papers on graph minors.

Definition 2.1 *A tree-decomposition of a graph $G = (V, E)$ is a pair (X, T), where $T(I, F)$ is a tree, and $X = \{X_i | i \in I\}$ is a family of subsets of V, one for each node of T, such that:*

- $\bigcup_{i \in I} X_i = V$.
- *for each edge $(u, v) \in E$, there exists an $i \in I$ such that $u, v \in X_i$.*
- *for all $i, j, k \in I$, if j is on the path from i to k in T, then $X_i \cap X_k \subseteq X_j$.*

The *treewidth* of a tree-decomposition (X, T) is $\max_{i \in I} |X_i| - 1$. The treewidth of a graph G is the minimum treewidth over all possible tree-decompositions of G.

A natural generalization of trees is the class of *k-trees*, which is defined recursively as follows:

- A complete graph on k vertices is a k-tree.
- If $G = (V, E)$ is a k-tree and $C \subseteq G$ is a clique of size k and $x \notin V$, then $H = (V \cup \{x\}, E \cup \{(u, x) | u \in C\})$ is a k-tree.

- There are no further k-trees.

A graph is called a *partial k-tree* if it is a subgraph of a k-tree. It can be shown [19] that every graph G has treewidth at most k if and only if it is a partial k-tree. Bodlaender [3] shows that for fixed k, there is a linear time algorithm that determines whether a given graph G is a partial k-tree, and if so, outputs a tree-decomposition of G with treewidth at most k.

We say a tree-decomposition (X, T) of treewidth k is *smooth*, if $T$is a binary tree and:

- for all $i \in I$: $|X_i| = k + 1$,
- every internal node X_i has two children, and if X_L and X_R are its children, then either $X_L = X_i$ or $X_R = X_i$,
- for all $(i, j) \in F$: $k \leq |X_i \cap X_j| \leq k + 1$, and
- for each edge $(u, v) \in E$, there is at least one leaf $i \in I$, with $u, v \in X_i$.

It can be shown that any tree-decomposition of a graph G can be transformed into a smooth tree-decomposition of G with the same treewidth and size $O(n)$ in linear time. (see [2] and [3]). For the purpose of our algorithms, we assume that along with the input graph we are given a smooth tree-decomposition of it.

For a given graph G with treewidth at most k, let (X, T) be a smooth tree-decomposition of it. For each node X_i of T we define T_i to be the subtree of T rooted at i. We define V_i and E_i to be the vertex set and edge set of the graph induced by the vertices that are in some node of T_i. More precisely, if X_i is a leaf then $V_i = X_i$ and $E_i = \{(u, v) \in E | u, v \in X_i\}$; if X_i is an internal node with children X_L and X_R then $V_i = V_L \cup V_R$ and $E_i = E_L \cup E_R$. We denote by G_i the graph with vertex set V_i and edge set E_i. Note that by these definitions, the only common vertices of $G_L = (V_L, E_L)$ and $G_R = (V_R, E_R)$ are in X_i.

For the purpose of the algorithm, we need to define another set of edges. We know that for every edge $e = (u, v) \in E$, there is at least one leaf i of tree T such $u, v \in X_i$. We fix one such i as the *representative* of edge e, and denote it by $\mathrm{rep}(e)$. If X_i is a leaf in T then E'_i is defined to be the set of edges of the graph whose representative is i, and if X_i has children X_L and X_R then $E'_i = E'_L \cup E'_R$. Equivalently, $E'_i = \{e \in E | \mathrm{rep}(e) \in T_i\}$. Note that by this definition, for every node X_i: $E'_i \subseteq E_i$, and if X_i has two children X_L and X_R then $E'_L \cap E'_R = \emptyset$. We denote the graph on vertex set V_i and edge set E'_i by G'_i.

3 The Strong Edge Coloring Algorithm

Here we prove the main result of this paper:

Theorem 3.1 *For every fixed integer k, there is a deterministic algorithm that, given a graph G with treewidth k on n vertices and an integer s, determines in time $O\left(n(s+1)^{2^{4(k+1)+1}}\right)$ whether G has a strong edge coloring using at most s colors or not, and if so finds such a strong edge coloring.*

We give a partial proof of this theorem by describing an algorithm to determine whether a strong edge coloring of G with at most s colors exists or not. This dynamic programming algorithm, which is easier to describe, can be easily modified (by storing some extra information in the tables) so that it finds a strong edge coloring with s colors, if one exists.

Let (X,T) be a smooth tree-decomposition of G and let $C = \{1, 2, \ldots, s\}$ be the set of colors. For a node X_i of T, a mapping $f : E'_i \longrightarrow C$ is called a *partial coloring of* G'_i. A partial coloring of G'_i is *proper* if it is a strong edge coloring of G'_i with respect to the edges in G_i. More precisely, mapping f is proper if for every two edges $e, e' \in E'_i$, if e and e' have a common endpoint or there is an edge $e'' \in E_i$ (and not necessarily in E'_i), then $f(e) \neq f(e')$. A proper partial coloring is *extendible*, if it can be extended to a strong edge coloring of G. Consider a partial coloring f of G'_i. For a vertex $v \in V_i$, we define $L_f(v)$ to be the set of colors of the edges in E'_i incident with v, and $L^2_f(v)$ to be the set of colors of the edges in E'_i that are incident with vertices that are adjacent to v by edges in E_i. We also define $N_f(X_i, c) \subseteq X_i$ and $N^2_f(X_i, c) \subseteq X_i$ as follows:

$$N_f(X_i, c) = \{v \in X_i : c \in L_f(v)\},$$
$$N^2_f(X_i, c) = \{v \in X_i : c \in L^2_f(v) - L_f(v)\}.$$

For each pair $A, B \subseteq X_i$, we define $CL_f(A, B) \subseteq C$, the *color class function of* f *on* X_i, as:

$$CL_f(A, B) = \{c \in C : A = N_f(X_i, c), B = N^2_f(X_i, c)\}.$$

From this definition, we immediately have:

Observation 3.2 *Let f be a partial coloring of G'_i and let CL_f be the color class function of f. Then $\{CL_f(A, B) : A, B \subseteq X_i\}$ is a partitioning of C.*

Let $\mathcal{P}(X_i)$ be the power set of X_i. A mapping $PC : \mathcal{P}(X_i) \times \mathcal{P}(X_i) \longrightarrow C$ is called a *pair count* of X_i if there exists a partial coloring f of G'_i, such that:

$$\forall A, B \in \mathcal{P}(X_i) : PC(A, B) = |CL_f(A, B)|.$$

In this case, PC is called the pair count of the partial coloring f. By this definition and observation 3.2 it follows that, for a pair count PC of X_i:

$$\sum_{A,B \subseteq X_i} PC(A, B) = s.$$

A pair count PC of X_i is *active* if it is the pair count of a proper partial coloring. The following lemma shows that the active pair counts can be seen as equivalence classes of proper partial colorings. So, instead of considering all possible proper partial colorings of G'_i, it is enough to consider only different active pair counts of node X_i.

Lemma 3.3 *Let f and g be two proper partial colorings of G'_i that have the same pair count. Then f is extendible if and only if g is extendible.*

Proof: Suppose that f is extendible. Since f and g have the same pair count, there is a relabeling of the colors $\pi : C \longrightarrow C$, such that for all $e \in E'_i$ that are adjacent to a vertex of X_i or to a neighbor of a vertex of X_i: $f(e) = \pi(g(e))$. This implies that the coloring $\pi \circ g$ can be extended to a strong edge coloring h of G: let $h(e) = \pi(g(e))$ for $e \in E'_i$, and $h(e) = f(e)$ otherwise. Therefore, $\pi^{-1} \circ h$ is a strong edge coloring of G that extends g. Thus, g is extendible. By symmetry, the same arguments work if we assume that g is extendible. ∎

So the only information needed to determine whether a proper partial coloring is extendible or not is its pair count.

For each node X_i of T, we compute a table of all active pair counts of that node. Since $|X_i| = k+1$ there are at most $(s+1)^{2^{2(k+1)}}$ active pair counts for X_i, which is polynomial in the input size. We compute the table of an internal node after computing the tables of its two children. The tables of the leaves can be computed directly, as they don't have any children. The goal is to compute the table of active pair counts of X_r, the root of T. Since $G_r = G$ and $E'_r = E$, a proper partial coloring of G'_r is, by definition, a strong edge coloring of G. Thus, G has a strong edge coloring with s colors, if and only if the table computed for X_r is not empty.

For a leaf X_j, we enumerate all proper partial colorings $f : E'_j \longrightarrow \min(s, |E'_j|)$, and compute the pair count of each and store it in a table, if it is not already in the table. Since $|E'_j| \leq \frac{(k+1)(k+2)}{2}$, the number of distinct partial colorings of E'_j is $O(1)$, and so is the running time of computing the table of X_j.

Now let's see how to compute the table of active pair counts of an internal node X_i, after computing the table of its children. Suppose that X_L and X_R are the two children of X_i. Recall that T is a smooth tree-decomposition, and therefore, $X_L \cap X_R = k$. Without loss of generality, assume that $X_L = X_i$, and let $X_L = \{v_0, v_1, \ldots, v_k\}$ and $X_R = \{v_1, v_2, \ldots, v_{k+1}\}$. If f is a partial coloring of G'_i, then by $f|G'_L$ and $f|G'_R$ we mean the restriction of f to E'_L and E'_R, respectively. We call a mapping $QC : \mathcal{P}(X_L) \times \mathcal{P}(X_L) \times \mathcal{P}(X_R) \times \mathcal{P}(X_R) \longrightarrow C$ a *quad count* on X_i if there exists a partial coloring f of G'_i, such that for each quadruple (A_L, B_L, A_R, B_R) with $A_L, B_L \subseteq X_L$ and $A_R, B_R \subseteq X_R$:

$$QC(A_L, B_L, A_R, B_R) = |CL_{f_L}(A_L, B_L) \cap \ CL_{f_R}(A_R, B_R)|,$$

where $f_L = f|G'_L$ and $f_R = f|G'_R$. This quad count is called the quad count of partial coloring f. A quad count QC of X_i is *active* if it is the quad count of a proper partial coloring. The following lemma shows how to compute the set of all active quad counts of X_i, knowing the set of all active pair counts of X_L and X_R:

Lemma 3.4 *A quad count QC of X_i is active if and only if it satisfies conditions (a) and (b) defined below:*

(a) If $QC(A_L, B_L, A_R, B_R) > 0$ then

- *(a.1) $A_L \cap A_R = \emptyset$.*
- *(a.2) $A_L \cap B_R = \emptyset$.*
- *(a.3) $A_R \cap B_L = \emptyset$.*
- *(a.4) $\forall u, v \in A_L \cup A_R : (u, v) \notin E_i$.*

(b) There exist active pair counts PC_L and PC_R of X_L and X_R, respectively, such that:

- *(b.1) For each pair $A_L, B_L \subseteq X_L$:*

$$PC_L(A_L, B_L) = \sum_{A,B \subseteq X_R} QC(A_L, B_L, A, B).$$

(b.2) For each pair $A_R, B_R \subseteq X_R$:

$$PC_R(A_R, B_R) = \sum_{A,B \subseteq X_L} QC(A, B, A_R, B_R).$$

Proof: *Necessity:* Assume that QC is an active quad count and f is a proper partial coloring whose quad count is QC. Let $f_L = f|G'_L$ and $f_R = f|G'_R$. It is easy to see that the pair counts of f_L and f_R are equal to PC_L and PC_R as defined in conditions (b.1) and (b.2), respectively. Therefore, PC_L and PC_R are active pair counts.

Now assume that $QC(A_L, B_L, A_R, B_R) > 0$, for some $A_L, B_L \subseteq X_L$ and $A_R, B_R \subseteq X_R$. From the definition of a quad count, it follows that if $v \in A_L \cap A_R$ then there are two distinct edges incident with v, one in E'_L and one in E'_R, that have the same color in f, contradicting that f is a proper partial coloring. We get similar contradictions if any of conditions (a.2) to (a.4) are violated.

Sufficiency: Assume that QC is a quad count of X_i that satisfies conditions (a) and (b). We want to show that QC is active. Let f be a partial coloring whose quad count is QC and define $f_L = f|G'_L$, $f_R = f|G'_R$, and let PC_{f_L} and PC_{f_R} be the pair counts of f_L and f_R, respectively. Suppose that h_L and h_R are two proper partial colorings of G'_L and G'_R, whose pair counts are equal to PC_L and PC_R, respectively. By definitions of f_L, f_R, and conditions (b.1) and (b.2):

$$\forall A_L, B_L \subseteq X_L : PC_L(A_L, B_L) = PC_{f_L}(A_L, B_L),$$

and

$$\forall A_R, B_R \subseteq X_R : PC_R(A_R, B_R) = PC_{f_R}(A_R, B_R).$$

It is not difficult to see that there exist two relabelings of the colors, $\pi_L : C \longrightarrow C$ and $\pi_R : C \longrightarrow C$, such that the color class functions of $\pi_L \circ h_L$ and f_L are equal, and similarly for $\pi_R \circ h_R$ and f_R. That is:

$$\forall A_L, B_L \subseteq X_L : CL_{\pi_L \circ h_L}(A_L, B_L) = CL_{f_L}(A_L, B_L), \tag{1}$$

$$\forall A_R, B_R \subseteq X_R : CL_{\pi_R \circ h_R}(A_R, B_R) = CL_{f_R}(A_R, B_R). \tag{2}$$

Now define the coloring $f' : E'_i \longrightarrow C$ as follows: $f'(e) = \pi_L \circ h_L(e)$, if $e \in E'_L$, and $f'(e) = \pi_R \circ h_R(e)$ if $e \in E'_R$. By this definition and Equations (1) and (2), f' has the same quad count as f does, which is QC. We show that f' is a proper partial coloring, and therefore, QC is an active quad count. Note that $\pi_L \circ h_L$ and $\pi_R \circ h_R$ are two proper partial colorings. By way of contradiction, assume that f' is not proper.

First consider the case that $e_1 = (u, v)$ and $e_2 = (u', v)$ are two distinct edges of E'_i with $f'(e_1) = f'(e_2)$. Since e_1 and e_2 cannot be both in E'_L or E'_R (because $\pi_L \circ h_L$ and $\pi_R \circ h_R$ are both proper), without loss of generality, we assume that $e_1 \in E'_L$ and $e_2 \in E'_R$. Therefore, $v \in X_L \cap X_R$, which implies that there exist subsets $A_L, B_L \subseteq X_L$ and $A_R, B_R \subseteq X_R$, for which $v \in A_L \cap A_R$ and $QC(A_L, B_L, A_R, B_R) \geq 1$ (because of $f'(e_1)$). But this contradicts condition (a.1).

The other possibility for f' to be an improper partial coloring is when there are three edges $e_1 = (u_1, v)$, $e_2 = (u_2, v')$, and $e_3 = (v, v')$, where $e_1, e_2 \in E'_i$ and $e_3 \in E_i$, such that $f'(e_1) = f'(e_2)$. Since $\pi_L \circ h_L$ is proper, e_1 and e_2 cannot be both in E'_L. Otherwise, $e_3 \notin E_L$ which implies that $v, v' \notin V_L$, and therefore $e_1, e_2 \notin E'_L$, a contradiction. Similar arguments show that e_1 and e_2 cannot be both in E'_R. So let's assume that $e_1 \in E'_L$ and $e_2 \in E'_R$. This means that $v \in V_L$ and $v' \in V_R$. Using the fact that $E_i = E_L \cup E_R$: $e_3 \in E_L$ or $e_3 \in E_R$. Consider the case that $e_3 \in E_R$. This implies that

$v \in V_L \cap V_R$, i.e. $v \in X_L \cap X_R$. Therefore, there exist subsets $A_L, B_L \subseteq X_L$ and $A_R, B_R \subseteq X_R$, for which $QC(A_L, B_L, A_R, B_R) \geq 1$ (because of $f'(e_1)$) and, either $v \in A_L \cap B_R$ or $v, v' \in A_L \cup A_R$. The case that $v \in A_L \cap B_R$ contradicts condition (a.2) and the second case, together with the fact that $e_3 = (v, v') \in E_i$, contradicts (a.4). If we consider the case that $e_3 \in E_L$ similar arguments contradict conditions (a.3) or (a.4).

Thus, there are no two edges of the same color in f' that are either adjacent or connected to an edge from E_i. This completes the proof that f' is a proper partial coloring of G_i'. ■

To compute the set of all active quad counts of X_i, first we consider all (not necessarily active) quad counts of X_i. There are at most $(s+1)^{2^{4(k+1)}}$ quad counts QC on X_i. For each quad count QC, we determine whether QC is active or not by checking conditions (a) and (b) in Lemma 3.4. Checking condition (a) takes constant time, as $|A_L|$, $|B_L|$, $|A_R|$, and $|B_R|$ are all bounded by $k+1$. Condition (b) can be checked in time $O((s+1)^{2^{2(k+1)+1}})$ because there are at most $\left((s+1)^{2^{2(k+1)}}\right)^2$ pairs of PC_L and PC_R. Since the number of quad counts on X_i is at most $(s+1)^{2^{4(k+1)}}$, all active quad counts on X_i can be computed in time

$$O\left((s+1)^{2^{2(k+1)+1}}(s+1)^{2^{4(k+1)}}\right) = O\left((s+1)^{2^{4(k+1)+1}}\right).$$

We will show how to compute all active pair counts of X_i having all active quad counts of X_i. To do so, we need the following lemma.

Lemma 3.5 *Assume that f is a partial coloring of G_i', $f_L = f|G_L'$, and $f_R = f|G_R'$. Let CL_f, CL_{f_L}, and CL_{f_R} be the color class functions of f, f_L, and f_R, respectively. Then for all $A, B \subseteq X_i$:*

$$CL_f(A,B) = \bigcup \left(CL_{f_L}(A_L, B_L)\ \cap CL_{f_R}(A_R, B_R)\right), \tag{3}$$

where the union is taken over all quadruples (A_L, B_L, A_R, B_R) satisfying:

$$A = A_L \cup A_R - \{v_{k+1}\}, \tag{4}$$

$$B = B' \cup B_L \cup B_R - \{v_{k+1}\}, \tag{5}$$

where

$$B' = \{v \in X_i - A : \exists u \in A_L \cup A_R \ s.t.\ (u,v) \in E_i\}.$$

Proof: First consider some color $c \in CL_f(A,B)$. We want to show it belongs to the right-hand-side of (3). Since CL_{f_L} and CL_{f_R} are each a partitioning of C, therefore, c belongs to exactly one $CL_{f_L}(A_L, B_L)$ and exactly one $CL_{f_R}(A_R, B_R)$. By definition of f_L and f_R: $A_L \subseteq A$, $(A_R - \{v_{k+1}\}) \subseteq A$, $B_L \subseteq B$, and $(B_R - \{v_{k+1}\}) \subseteq B$. To complete the proof of this part we prove that $A \subseteq A_L \cup A_R - \{v_{k+1}\}$ and $B \subseteq B' \cup B_L \cup B_R - \{v_{k+1}\}$. For each vertex $v \in X_i$, which is incident with a colored edge e (i.e. $v \in A$), since $E_i' = E_L' \cup E_R'$: $e \in E_L'$ or $e \in E_R'$. Therefore, $v \in A_L$ or $v \in A_R$, which implies $A \subseteq A_L \cup A_R - \{v_{k+1}\}$. For each vertex $v \in X_i$ which is adjacent (by an edge $(v,u) \in E_i$) to a colored edge e (i.e. $v \in B$), either $e \in E_L'$ or $e \in E_R'$. If $e \in E_L'$ and $(v,u) \in E_L$ then clearly $v \in B_L$. If $e \in E_R'$ and $(v,u) \in E_R$ then $v \in B_R$. Otherwise, let's assume $e \in E_L'$ and $(v,u) \in E_i - E_L$ (a similar argument works for the case that $e \in E_R'$ and $(v,u) \in E_i - E_R$). This means that $u \in A_L$ and therefore, $v \in B'$. So in all cases $v \in B' \cup B_L \cup B_R - \{v_{k+1}\}$, which implies $B \subseteq B' \cup B_L \cup B_R - \{v_{k+1}\}$. Thus, c belongs to the right-hand-side of Equation (3).

Now assume that c is a color in the right-hand-side of (3). This means that there exists some quadruple (A_L, B_L, A_R, B_R) satisfying Equations (4) and (5), such that $c \in (CL_{f_L}(A_L, B_L) \cap CL_{f_R}(A_R, B_R))$. So the vertices in X_i that are incident with an edge of E'_L with color c are those in A_L, and the vertices of X_i that are incident with an edge of E'_R with color c are those in $A_R - \{v_{k+1}\}$. Therefore, $N_f(X_i, c) = A_L \cup A_R - \{v_{k+1}\}$. Similarly, it is not difficult to see that the vertices of X_i that are *not* incident with color c but one of their neighbors is incident with color c, are precisely those that are in $B' \cup B_L \cup B_R - \{v_{k+1}\}$, which means that $N^2_f(X_i, c) = B$. Thus $c \in CL_f(A, B)$. ∎

Lemma 3.6 *A pair count PC of X_i is active if and only if there exists an active quad count QC of X_i such that for each pair $A, B \subseteq X_i$:*

$$PC(A, B) = \sum QC(A_L, B_L, A_R, B_R), \tag{6}$$

where the summation is taken over all quadruples (A_L, B_L, A_R, B_R) satisfying Equations (4) and (5).

Proof: *Necessity:* Assume that PC is an active pair count of X_i and f is a proper partial coloring of G'_i whose pair count is PC. Let QC be the quad count of f. Therefore, QC is active. We show that PC and QC satisfy Equation (6). Define $f_L = f|G'_L$ and $f_R = f|G'_R$, and let CL_f be the color class function of f on X_i. Similarly, define CL_{f_L} and CL_{f_R}. By definition of a quad count:

$$QC(A_L, B_L, A_R, B_R) = |CL_{f_L}(A_L, B_L) \ \cap \ CL_{f_R}(A_R, B_R)|. \tag{7}$$

An important observation at this point is that

$$\{CL_{f_L}(A_L, B_L) \ \cap \ CL_{f_R}(A_R, B_R) : A_L, B_L \subseteq X_L \text{ and } A_R, B_R \subseteq X_R\}$$

is a partition of C. Therefore:

$$\left|\bigcup (CL_{f_L}(A_L, B_L) \ \cap \ CL_{f_R}(A_R, B_R))\right| = \sum |(CL_{f_L}(A_L, B_L) \ \cap \ CL_{f_R}(A_R, B_R))|, \tag{8}$$

where the union and summation are taken over all quadruples that satisfy (4) and (5). Using (3), (7), (8), and definition of an active pair count it follows that PC and QC satisfy (6).

Sufficiency: Assume that QC is an active quad count on X_i, and f is a proper partial coloring of G'_i whose quad count is QC. Define f_L, f_R, CL_f, CL_{f_L}, and CL_{f_R} as in the previous case. Then by Lemma 3.5, Equation (6), and definition of a quad count:

$$\begin{aligned} |CL_f(A, B)| &= \left|\bigcup (CL_{f_L}(A_L, B_L) \ \cap \ CL_{f_R}(A_R, B_R))\right| \\ &= \sum |(CL_{f_L}(A_L, B_L) \ \cap \ CL_{f_R}(A_R, B_R))| \\ &= \sum QC(A_L, B_L, A_R, B_R) \\ &= PC(A, B), \end{aligned}$$

where the union and summations are taken over all quadruples that satisfy (4) and (5). Therefore, the mapping PC defined as in Equation (6) would be an active pair count. ∎

Using Lemma 3.6 we compute all active pair counts of X_i from active quad counts of X_i. There are at most $(s+1)^{2^{4(k+1)}}$ different quad count QC of X_i, and for each QC we can compute the corresponding pair count satisfying Equation (6) in $O(1)$. Therefore, having all the active quad counts of X_i, we can compute all the active pair counts of X_i in time $O((s+1)^{2^{4(k+1)}})$.

Overall, since the number of nodes of T is $O(n)$, we can compute all the tables of all the nodes (including the root) of T, in time

$$O\left(n(s+1)^{2^{4(k+1)+1}}\right).$$

This completes the proof of Theorem 3.1.

4 Concluding Remarks

In this paper, we gave the first polynomial time algorithm for the strong edge coloring problem on partial k-trees, for fixed k. We didn't describe how to find a strong edge coloring using the specified number of colors. But this can be done, by doing slight modifications in the algorithm (using extra bookkeeping). We skip the details.

Although the running time of the given algorithm is polynomial, it is doubly exponential in k, like many other algorithms for different problems on graphs with bounded treewidth. Hence, it remains an open problem to find an algorithm with a better running time. Perhaps the technique used for finding linear time algorithms for standard edge-coloring [23] and total coloring [12] of partial k-trees might be helpful here. However, it is worth to mention that, this algorithm can be parallelized, to be run in $O(\log n)$ time on a CRCW PRAM, using the same techniques as in Bodlaender and Hagerup [4], and Zhou et al. [22].

Acknowledgments: The author would like to thank the referees for their helpful comments and suggestions.

References

[1] S. Arnborg and J. Lagergren, *Easy problems for tree-decomposable graphs*, J. of Algorithms 12 (1991), 308-340.

[2] H.L. Bodlaender, *Polynomial Algorithms for Graph Isomorphism and Chromatic Index on Partial k-trees*, J. of Algorithms 11 (1990) 631-643.

[3] H.L. Bodlaender, *A linear time algorithm for finding tree-decompositions of small treewidth*, SIAM J. on Computing 25 (1996) 1305-1317.

[4] H.L. Bodlaender and T. Hagerup, *Parallel algorithms with optimal speedup for bounded treewidth'*, SIAM J. on Computing 27 (1998) 1725-1746.

[5] R.B. Borie, R.G. Parker, and C.A. Tovey, *Automatic generation of linear-time algorithms from predicate calculus description of problems on recursively constructed graph families*, Algorithmica 7 (1992), 555-581.

[6] K. Cameron, *Induced Matchings*, Disc. Appl. Math. 24 (1989) 97-102.

[7] R. J. Faudree, A. Gyárfás, R. H. Schelp, and Zs. Tuza, *Induced matchings in bipartite graphs*, Discrete Math. 78 (1989) 83-87.

[8] R. J. Faudree, A. Gyárfás, R.H. Schelp, and Zs. Tuza, *The strong chromatic index of graphs*, Ars Combinatorica 29B (1990) 205-211.

[9] G. Fricke and R. Laskar, *Strong Matchings on Trees*, Congr. Numer. 89 (1992) 239-243.

[10] M.C. Golumbic and R. Laskar, *Irredundancy in circular arc graphs*, Disc. Appl. Math. 44 (1993) 79-89.

[11] M.C. Golumbic and M. Lewenstein, *New results on induced matchings*, Disc. Appl. Math. 101 (2000) 157-165.

[12] S. Isobe, X Zhou, and T. Nishizeki, *A linear Algorithm for Finding Total Colorings of Partial k-Trees*, Proc. of ISAAC'99, LNCS 1741, 347-356.

[13] C.W. Ko and F.B. Shepherd, *Adding an identity to a totally unimodular matrix*, London School of Economics Operations Research Working Paper, LSEOR 94.14, July 1994.

[14] M. Mahdian, *On the computational complexity of strong edge coloring*, Disc. Appl. Math. 118, pp. 239–248, 2002.

[15] S.T. McCormick, *Optimal approximation of sparse Hessians and its equivalence to a graph coloring problem*, Mathematical Programming 26 (1983), 153-171.

[16] N. Robertson and P.D. Seymour *Graph Minors II: Algorithmic aspects of tree-width*, J. of Algorithms 7 (1986) 309-322.

[17] N. Robertson and P.D. Seymour *Graph Minors V: Excluding a planar graph*, J. Comb. Theory Series B 41 (1986) 92-114.

[18] N. Robertson and P.D. Seymour *Graph Minors IV: Tree-width and well quasi-ordering*, J. Comb. Theory Series B 48 (1990) 227-254.

[19] P.Scheffler *Die Baumweite von Graphen als ein Mab fur die Komplizierthit algorithmischer Probleme*, PH.D thesis, Akademie der Wissenschafte der DDR, Berlin, 1989.

[20] L.J. Stockmeyer and V.V. Vazirani, *NP-completeness of some generalizations of the maximum matching problem*, Inform. Proc. Letters 15(1) 1982 14-19.

[21] R. Sritharan and Y. Tang, *Finding a maximum induced matching in weakly chordal graphs*, A talk presented by Y. Tang in "Horizons in Combinatorics/16th Shanks Lecture Series", May 21-24, 2001 Vanderbilt University, Nashville, TN, USA.

[22] X. Zhou, Y. Kanari, T. Nishizeki, *Generalized Vertex-Colorings of partial k-trees*, IEICE transactions, E83-A 4 (2000), 671-677.

[23] X. Zhou, S. Nakano, and T. Nishizeki, *Edge-coloring partial k-trees*, J. of Algorithms 21 (1996) 598-617.