Lecture 32: NP-Completeness

Agenda:

- Classes P, NP (recall), and co-NP
- The notions of hardest and complete
- Polynomial-time reduction
- NP-completeness proof steps

Reading:

- Textbook pages 979 – 995

The classes of P and NP (recall):

- Class P
  - decision problem
  - there exists some algorithm solving the problem in polynomial time
  - which of the above problems are in P:
    1. sorting
    2. longest common subsequence
    3. minimum spanning tree
    4. single-source shortest paths
    5. shortest x-to-y path
    6. determining Eulerian graphs

- Class NP
  - decision problem
  - for every **yes**-instance, there exists a proof that the answer is yes and the proof can be verified in polynomial time
  - which of the above problems are in NP:
    1. sorting
    2. longest common subsequence
    3. minimum spanning tree
    4. single-source shortest paths
    5. shortest x-to-y path
    6. determining Eulerian graphs
    7. longest x-to-y path
    8. determining Hamiltonian graphs

P, NP, and co-NP:

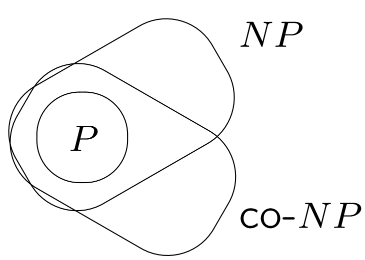
Question: is "determining if a graph is NOT Hamiltonian" in NP?

What is a proof to answer **yes**?

- Class co-NP
  - decision problem
  - for every **no**-instance, there exists a proof that the answer is no and the proof can be verified in polynomial time
  - which of the above problems are in co-NP:
    1. sorting
    2. longest common subsequence
    3. minimum spanning tree
    4. single-source shortest paths
    5. shortest x-to-y path
    6. determining Eulerian graphs
    7. determining non-Hamiltonian graphs

  One more example: "is this number prime?"

- The relationships among P, NP, and co-NP



Some well-known problems in NP:

- k-clique:

  Given a graph, does it have a size k clique?

- k-independent set:

  Given a graph, does it have a size k independent set?

- k-coloring:

  Given a graph, can the vertices be colored with k colors such that adjacent vertices get different colors?

- Satisfiability (SAT):

  Given a boolean expression, is there an assignment of truth values (T or F) to variables such that the expression is satisfied (evaluated to T)?

- Travel salesman problem (TSP):

  Given an edge-weighted complete graph and an integer ℓ, is there a Hamiltonian cycle of length at most ℓ?

- $\vdots$

Completeness and Reduction:

- For the class NP of problems, which one is the hardest?

  Reasons to ask:

  - Is every problem in NP solvable in polynomial time?
  - If not, what are the characteristics of the hard problems?

- Hardest problem — solving it means you can solve every other in NP

- Informally, complete = hardest;

  Formally, we need "polynomial-time reduction":

  - decision problem Π_1 is polynomial-time reducible to Π_2, written as
  $$\Pi_1 \leq_p \Pi_2,$$
  if
  - $\exists$ polynomial-time transformation function t which
  - maps instances of Π_1 to instances of Π_2, such that
  - for every instance x of Π_1, the answer to x is the same as the answer to $t(x)$

- Formally, $\Pi \in NP$ is NP-complete if every other problem Π' is polynomial-time reducible to Π: $\Pi' \leq_p \Pi$

Examples of (polynomial-time) reduction:

1. k-independent set is reducible to k-clique

   `Proof.` From a k-independent set instance (G, k), construct a k-clique instance to be $(\overline{G}, k)$, where $\overline{G}$ is the complement graph of G.

   The construction takes $\Theta(n^2)$ time for every instance (G, k) containing n vertices and thus polynomial --- take this construction as the transformation function.

   Now, X is an independent set in G iff it is a clique in $\overline{G}$. Proof done.

2. 3-SAT is reducible to SAT

   A trivial exercise.

3. SAT is reducible to 3-SAT

   A non-trivial exercise.

Proof of NP-completeness:

- Suppose we want to prove Π is NP-complete, then we need to prove
  1. $\Pi \in NP$
  2. for every problem $\Pi' \in NP$, Π' is reducible to Π
- Question:
  1. How many problems in NP? — ∞ ??!! inefficient way

Proof of NP-completeness:

- Observations:
  - $\Pi \in NP$
  - Π' is NP-complete and is reducible to Π
  - $\Longrightarrow \Pi$ is NP-complete !
- Why it is true:
  - for every problem $\Pi'' \in NP$
  - since Π' is NP-complete, Π'' is reducible to Π'
  - using the transitivity of reduction
  - Π'' is reducible to Π
- Switching our goal to:
  Prove that $\Pi \in NP$
  Look for an NP-complete problem Π' and
  Prove that Π' is reducible to Π
- Where to start with — we need a first NP-complete problem

  Cook's Theorem: SAT is NP-complete.

Have you understood the lecture contents?

well	ok	not-at-all	topic
□	□	□	classes P, NP, co-NP
□	□	□	relationships among the 3 classes
□	□	□	NP-complete
□	□	□	polynomial time reduction
□	□	□	steps of NP-completeness proof