CMPUT 675: Approximation Algorithms **Fall 2011**

Lecture 16 (Nov 1, 2011): Approximation of metrics by Tree metrics

Lecturer: Mohammad R. Salavatipour *Scribe: Amritpal Saini*

16.1 Introduction

A common and useful technique to simplify some problems given in a metric space (V, d) is to approximate the metric with a simpler metric. An exact embedding of a metric space (V, d) into another metric space (V', d') is a mapping $f : V \to V'$ such that for all $x, y \in V$, $d(x, y) = d'(f(x), f(y))$. Typically we cannot get an exact embedding and we settle for embeddings that preserve the distances approximately.

Tree metrics have proved to be a very useful tool for deriving approximation algorithms for several problems on graphs. The idea is that we embedd a metric graph into a tree and we use the distances between the points in the tree to approximate the distances of the corresponding vertices in the original graph. A tree metric (V', T) approximates distance metric d_G on a set of vertices V with $V \subseteq V'$ if for each pair of vertices $u, v \in V$ $d_G(u, v) \leq d_T(u, v) \leq \alpha d_G(u, v)$ for some value of α. α is known as the distortion of the embedding of d_G in the tree metric. Since the distances on the tree are a relatively good approximation of the distances of the nodes in the original graph, we can typically get a good approximation for the problem on hand if we can solve the problem exactly (or with a good approximation) on the tree. Unfortunately, we cannot always embed a graph into a tree with low distortion. For example, it is easy to prove that for C_n (a cycle on n vertices) embedding it to any tree metric has distortion at least $\Omega(n)$. However, if we produce a tree randomly, we can show that the expected distances of the nodes in the tree are not much bigger than their distances in the original graph. More specifically, we see a randomized algorithm for producing a tree T such that for $u, v \in V$, $d_G(u, v) \leq d_T(u, v)$ and $E[d_T(u, v)] \leq O(\log n) d_G(u, v)$.

16.2 Basic Idea

The basic idea behind the method of approximating graph metric with tree metric is hierarchical decomposition of the metric d. Let $\Delta = max_{u,v} d_G(u, v)$. We partition the graph G into pieces with small diameters such that no pair of $u, v \in V$ is spread with too high probability. We associate a tree with our decomposition. The nodes at each level of the tree corresponds to some partition of V. The root of the tree corresponds to V itself. Every leaf corresponds to a single vertex. A node at some level corresponds to some subset of the vertex set V. For a node in tree T that corresponds to a set S, the vertices in S will be vertices in a ball of radius less than 2^i and at least 2^{i-1} centered on some vertex. Each node of the tree will be a vertex in V'. For each edge in the tree the length of the edge joining children at level $i - 1$ to i is 2^i. Figure 16.1 depicts the tree constructed from hierarchical decomposition of the graph metric.

16.3 Algorithm

The randomized algorithm for generating a random tree metric is given in Figure 16.2.

Our goal in the rest of this lecture is to prove that the algorithm for tree metric indeed achieves $E[d_T(u, v)] \leq O(\log n) d_G(u, v)$.

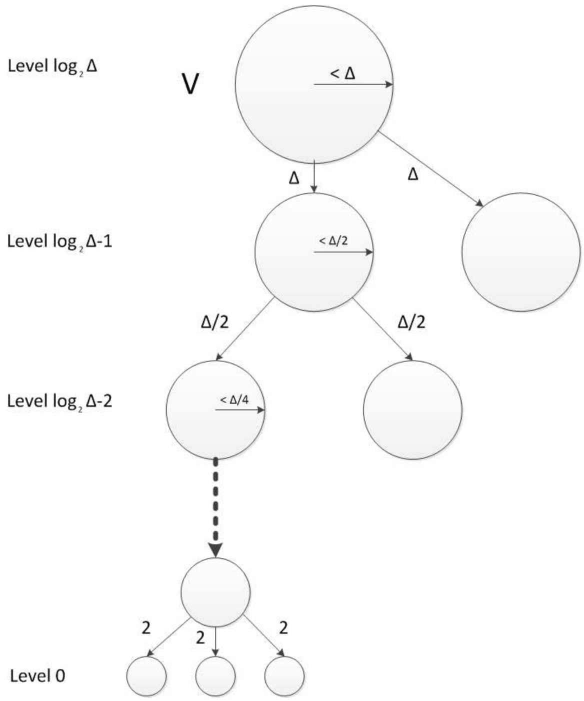

Figure 16.1: A hierarchical cut decomposition of the metric space. Image source: The Design of Approximation Algorithms by Williamsons and Shmoys

Lemma 1 *For any tree produced by hierarchical cut decomposition $d_G(u,v) \leq d_T(u,v)$ for all pairs of u, v. Also if the least common ancestor of u, v is at level i, then $d_T(u,v) \leq 2^{i+2}$.*

Proof. In the tree T for any pair of vertices u, v in a set S corresponding to a node in the tree, the distance is less than 2^{i+1}, since the radius of the ball corresponding to S is less than 2^i. Thus vertices u, v cannot be in a set at level $\lfloor \log(d_G(u,v)) \rfloor - 1$ or smaller as the distance between them would be less than $d_G(u,v)$. Thus the lowest level at which u, v can belong to same node is $\lfloor \log_2 d_G(u,v) \rfloor$. Thus, $d_T(u,v) \geq 2\sum_{j=1}^{\lfloor \log d_G(u,v)} 2^j \geq d_G(u,v)$. If the least common ancestor of u, v is at level i then $d_T(u,v) = 2\sum_{j=1}^{i} 2^j = 2^{i+2} - 4 \leq 2^{i+2}$. ∎

Definition: A vertex w settles the pair u, v on level i if w is the first vertex in the random permutation π of vertices such that at least one of u, v is in the ball $B(w, r_i)$.

Definition: A vertex w cuts the pair u, v on level i if exactly one of u, v is in the ball $B(w, r_i)$.

Lemma 2 *If X_{iw} be the event that w cuts (u,v) on level i then $\sum_{i=1}^{\log \Delta} \Pr[X_{iw}] \cdot 2^{i+3} \leq 16 \cdot d_G(u,v)$.*

Proof. Suppose $d_G(u,w) \leq d_G(v,w)$. Then the probability that w cuts (u,v) on level i is the probability that $u \in B(w, r_i)$ and $u \notin B(w, r_i)$ or that $d_G(u,w) \leq r_i < d_G(v,w)$. Now $r_i \in [2^{i-1}, 2^i)$ uniformly at random.

$$\Pr[X_{iw}] = \frac{\left|[2^{i-1}, 2^i) \cap [d_G(u,w), d_G(v,w))\right|}{|[2^{i-1}, 2^i)|} \tag{16.1}$$

$$= \frac{\left|[2^{i-1}, 2^i) \cap [d_G(u,w), d_G(v,w))\right|}{2^{i-1}} \tag{16.2}$$

(16.3)

Algorithm to create hierarchical tree decomposition

Pick a random permutation π of V
Set Δ to the smallest power of 2 greater than $2 \cdot max_{u,v} d_G(u,v)$
Pick $r_0 \in [1/2, 1)$ and set $r_i = 2^i r_0$ for $1 \le i \le \log_2 \Delta$
$\mathcal{C}(\log \Delta) = V$ be the root. Create a node corresponding to V.
for $i \leftarrow \Delta$ downto 1 do
    $\mathcal{C}(i-1) \leftarrow \emptyset$
    for all $C \in \mathcal{C}(i) do$
        $S \leftarrow C$
        for $j \leftarrow 1$ to n do
            if $B(\pi(j), r_{i-1}) \cap S \neq \emptyset$ then
                Add $B(\pi(j), r_{i-1}) \cap S$ to $\mathcal{C}(i-1)$
                Remove $B(\pi(j), r_{i-1}) \cap S$ from S
        Create tree nodes for sets in $\mathcal{C}(i-1)$.

Figure 16.2: Algorithm hierarchical tree decomposition

Then

$$2^{i+3}\Pr[X_{iw}] = \frac{2^{i+3}}{2^{i-1}} \left| \left[2^{i-1}, 2^i\right) \cap [d_G(u,w), d_G(v,w)) \right| \tag{16.4}$$

$$= 16 \left| \left[2^{i-1}, 2^i\right) \cap [d_G(u,w), d_G(v,w)) \right| \tag{16.5}$$

$$\tag{16.6}$$

Now the interval $\left[2^{i-1}, 2^i\right)$ for $i = 0$ to $\log_2 \Delta - 1$ partition the interval $[1/2, \Delta/2)$. Thus

$$\sum_{i=0}^{\log_2 \Delta - 1} 2^{i+3}\Pr[X_{iw}] \le 16 \left| [d_G(u,w), d_G(v,w)) \right| \tag{16.7}$$

$$= 16 \left(d_G(v,w) - d_G(u,w)\right) \tag{16.8}$$

$$\le 16 d_G(u,v) \tag{16.9}$$

■

Lemma 3 *If w is the j'th closest node to u, v then* $\Pr[S_{iw} | X_{iw}] \le 1/j$

Proof. If event X_{iw} happens then either u is in the ball $B(w, r_i)$ or v. So in order for w to settle the pair u, v given that it cuts u, v on level i, it must come before all the closer vertices z in the random permutation of vertices. If w is the jth closest vertex to the vertex, it settles the pair u, v with probability $1/j$. Now for j, $1 \le j \le n$ there is some vertex w, jth closes to the pair u, v, thus

$$\sum_{w \in V} b_w = \sum_{j=1}^{n} 1/j \tag{16.10}$$

$$= O(\log n) \tag{16.11}$$

■

Theorem 1 *Given a distance metric (V, d), such that $d_G(u,v) \ge 1$ for all $u \neq v, u, v \in V$, there is a randomized , polynomial time algorithm that produces a tree metric (V', T), $V \subseteq V'$, such that for $u, v \in V$, $d_G(u,v) \le d_T(u,v)$ and $E[d_T(u,v)] \le O(\log n) d_G(u,v)$.*

Proof. Consider any vertex pair u, v. The first claim follows from Lemma 1. Now we prove that $E\left[d_T\left(u,v\right)\right] \leq O\left(\log n\right) d_G\left(u,v\right)$. By Lemma 1, if they have a common ancestor at level $i+1$ then $d_T\left(u,v\right)\ leq 2^{i+3}$. For this u, v must be in different sets at level i. So there must be some w such that exactly one of u and v must be in the set corresponding to the ball centered on w on level i. Now let X_{iw} be the event that w cuts (u,v) on level i, and let S_{iw} be the event that w settles (u,v) on level i. Therefore:

$$d_T(u,v) \leq \max_{i=0,\ldots,\log\Delta} 2^{i+3} \text{ where } \exists w \in V : X_{iw} \wedge S_{iw}$$

Thus the expected distance between vertex pair u, v in the tree metric is

$$\begin{aligned}
E\left[d_T\left(u,v\right)\right] &\leq \sum_{w\in V}\sum_{i=0}^{\log\Delta-1} \Pr\left[X_{iw}\wedge S_{iw}\right]\cdot 2^{i+3} && (16.12)\\
&= \sum_{w\in V}\sum_{i=0}^{\log\Delta-1} \Pr\left[S_{iw}|X_{iw}\right]\cdot \Pr\left[X_{iw}\right]\cdot 2^{i+3} && (16.13)\\
&\leq \sum_{w\in V} b_w \sum_{i=0}^{\log\Delta-1} \Pr\left[X_{iw}\right]\cdot 2^{i+3} && (16.14)\\
&\leq 16 d_G(u,v)\sum_{w\in V} b_w \text{ [from Lemma 2]} && (16.15)\\
&\leq 16 d_G(u,v) O\left(\log n\right) \text{ [from Lemma 3]} && (16.16)
\end{aligned}$$

■