CMPUT 675: Approximation Algorithms	Winter 2005
Lecture 13: Mar 2	
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In the next two lectures we will be discusing about Packing Disjoint Steiner trees which is mostly related to networking problems. All these problems are "packing" problems, whereas most of the problems we have consideredso far (e.g. vertex cover, set cover, multiway cut, multicut, etc) are covering problems. Generally speacking, packing problems tend to be more difficult than covering problems.

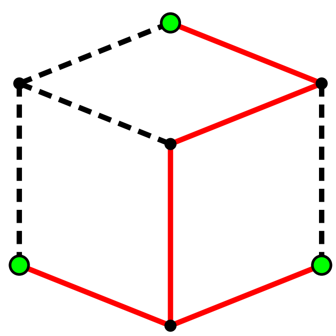
13.1 Packing Steiner Trees

Packing edge disjoint Steiner tree probelm is defined as follows:

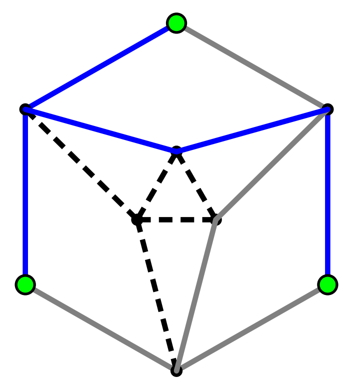
Input: An undirected graph $G(V,E)$ with a set $T \subseteq V$ of terminals; the vertices in $V - T$ are Steiner nodes.

Goal: Find the maximum number of edge-disjoint Steiner trees (each covering all the terminals).

This problem is NP-complete even if $|T| = 4$ or even if we are asked to find only two edge-disjoint Steiner trees.



(a) only 1 Steiner trees



(b) 2 edge-disjoint Steiner trees

Figure 13.1: Two examples. The green nodes are terminals

The problem can be generalized to a capacitated version. In this version we are also given a capacity function $C : E \to Z^*$, and we have to find maximum number of Steiner trees such that for every edge e, at most C_e selected trees contain e.

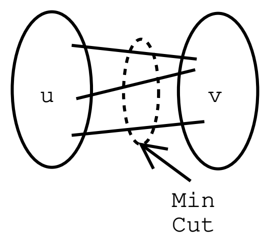
Observation 13.1 *We can assume that in any tree in a feasible solution, all the leaves are terminals (i.e. no Steiner point is a leaf).*

13.1.1 Special Cases

The following are two special cases that were explored in the past.

Speial Case 1: If $|T| = 2$, say $T = \{u, v\}$, then by Observation 13.1, any Steiner in the solution is a u, v-path. So we are asking for the maximum number of edge-disjoint u, v-paths. The well-known theorem of Menger shows that this is equal to the size of the minimum u, v-cut and we can comput it in polytime:

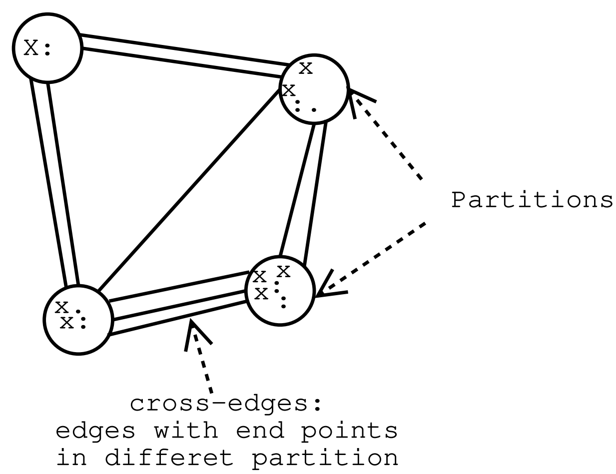
Theorem 13.2 (Menger) *The maximum number of edge-disjoint paths between u and v is equal to minimum number of edges whose removal disconnects u and v.*



Definition 13.2 For a set $T \subseteq V$, the T-edge connectivity is the minimum number of edges whose removal disconnects two vertices in T.

Special Case 2: If $T = V$, then we are looking for edge-disjoint spanning trees. This problem was studies in early sixties, and we have polynomial time algorithms that find maximum number of edge-disjoint spanning trees of a graph.

Consider a partition $\mathcal{P}$ of vertices of G such that every part has at least one Steiner Point.



The edges which run between different parts of $\mathcal{P}$ are called cross-edges.

Fact: Every Steiner tree must contain atleast $|\mathcal{P}| - 1$ cross edges.

Using this fact, an upper bound for the number of edge-disjoint Steiner tree is: $\frac{\text{number of cross-edgs}}{|\mathcal{P}|-1}$. This is true for all partitions. Therefore, the minimum of this ratio over all partitions is also an upper bound. Let edge-toughness of G be defined as $\min_{\mathcal{P}} \frac{number\ of\ cross-edges}{|\mathcal{P}|-1}$. Then:

Theorem 13.3 (Nash-Williams & Tutte '61) *The maximum number of edge-disjoint spanning trees of G is equal to the (integer part of) edge-toughness of G.*

We can actually find a largest set of edge-disjoint spanning trees in polynomial time using Edmond's matroid intersection algorithm. An immediate corollary of theorem of Nash-Williams and Tutte is:

Corollary 13.4 *Every $2k$-edge-connected graphs have k edge-disjoint spanning tree.*

Note that every $2k$-edge connected graph has at most $2k$ edge-disjoint spanning trees. Can we generalize Theorem 13.3 to Steiner trees? The answer is no! For example, the graph in Figure 1 has edge-toughness 2 but does not have 2-edge-disjoint Steiner trees. Kriesell made the following interesting conjecture:

Conjecture 13.5 (Kriesell) *If T is $2k$-edge connected then G contains k edge-disjoint Steiner trees.*

This conjecture is open even for $k = 2$. The following theorem of Lau shows that an approximate version of this conjecture is true.

Theorem 13.6 (Lau FOCS'04) *If T is $26k$-edge connected graph then G has k edge disjoint Steiner trees.*

Note that this is an approximate min-max theorem (since the edge-connectivity is clearly a lower bound for the number of edge-disjoint Steiner trees in a graph). Also, this theorem is constructive and shows how to find the trees. Therefore:

Corollary 13.7 *We get a 26-approximation algorithm for the problem of packing edge-disjoint Steiner trees.*

13.2 The Primal-Dual formulation

Let τ be the set of all Steiner Trees of G. We will have a 0/1 variable x_F for every Steiner tree $F \in \tau$. So there will be an exponential number of variables. First, we obtain an integer programming formulation of the problem and then derive its LP-relaxations.

$$\begin{array}{rcl} \textit{maximize} & \sum & x_F \\ \textit{subject to } \forall e \in E, & \sum\limits_{F: e \in F} & x_F \leq C_e \\ & & x_F = \{0, 1\} \end{array}$$

The LP-relaxation is obtained by replacing the constraint $x_F = \{0, 1\}$with $0 \leq x_F \leq 1$.

$$\begin{array}{rcl} \textit{maximize} & \sum & x_F \\ \textit{subject to } \forall e \in E, & \sum\limits_{F: e \in F} & x_F \leq C_e \\ & & x_F \geq 0 \end{array}$$

Next, we derive the dual problem. We have already seen how to find dual from the primal in earlier lectures.

$$
\begin{aligned}
\textit{minimize} \quad & \sum c_e y_e \\
\textit{subject to}\ \forall F \in \tau, \quad & \sum_{e \in F} y_e \geq 1 \\
& y_e \geq 0
\end{aligned}
$$

The dual LP can be interpreted as follows: assign weights (y_e's) to the edges such that the minimum weight Steiner tree has weight at least 1 and a linear function of the weights is minimized (for the case that all edge capacities are one we are to minimize the total weights of the edges).

Therefore, the separation oracle for the dual LP is the problem of finding a minimum weight Steiner tree. We know that this problem is NP-hard. Therefore, solving the dual LP (and so the primal as well) is NP-hard. But we can use a known approximation algorithm for minimum Steiner tree to solve these LPs approximately. In fact, the following interesting result holds:

Theorem 13.8 *There is an α- approximation algorithm for the maximum fractional Steiner tree packing if and only if there is an α- approximation for minimum weight Steiner tree problem.*

This theorem holds in a more general setting of of primal-dual LP's.