Load balancing, Balls & Bins

- n balls & n bins
- We throw each ball into a bin selected u.r. and independent of other balls
- Load of a bin: # of balls in it
- Applications: load balancing, hashing

Question 1: what is the probability of any two balls landing in the same bin? $1/n$

Question 2: what is the expected number of empty bins?

- let's first compute the probability of bin i being empty: b_i

$$\Pr[b_i] = \left(1 - \frac{1}{n}\right)^n \approx e^{-1}$$

Define $X_i = \begin{cases} 1 & \text{bin } i \text{ is empty} \\ 0 & \text{o.w.} \end{cases}$

$$X = \sum_{i=1}^{n} X_i, \qquad E[X] = E\left[\sum_i X_i\right] = \sum_i E[X_i] = n/e$$

Question 3: what is the probability of bin i having load k? load $\geq k$?

$\underbrace{\text{having load } k}_{\text{event } \mathcal{E}_i^k}$? $\underbrace{\text{load } \geq k}_{\text{event } \mathcal{E}_i^{\geq k}}$?

$$\Pr[\mathcal{E}_i^k] = \binom{n}{k}\left(\frac{1}{n}\right)^k\left(1-\frac{1}{n}\right)^{n-k}$$

$$\Pr[\mathcal{E}_i^{\geq k}] = \sum_{j=k}^{n}\binom{n}{j}\left(\frac{1}{n}\right)^j\left(1-\frac{1}{n}\right)^{n-j}$$

$$\leq \sum_{j=k}^{n}\binom{n}{j}\left(\frac{1}{n}\right)^j$$

$$\leq \sum_{j=k}^{n}\left(\frac{e\cancel{n}}{j}\right)^j\left(\frac{1}{\cancel{n}}\right)^j \quad \left(\frac{e}{j}\right)^j$$

$$\leq \left(\frac{e}{k}\right)^k\left(1+\frac{e}{k}+\left(\frac{e}{k}\right)^2+\cdots\right)$$

$$\leq \left(\frac{e}{k}\right)^k\left(\frac{1}{1-e/k}\right)$$

Question: what is the maximum load?

Theorem: The maximum load $\leq \dfrac{3\ln n}{\ln\ln n}$ with probability at least $1-\frac{2}{n}$.

Proof: choose $k = \dfrac{3\ln n}{\ln\ln n}$. Then

$$\Pr[\mathcal{E}_i^{\geq k}] \leq 2\left(\frac{e\ln\ln n}{3\ln n}\right)^{3\ln n/\ln\ln n}$$

$$\leq 2\exp\left[(1-\ln 3-\ln\ln n+\ln\ln\ln n)\,3\ln n/\ln\ln n\right]$$

$$\leq 2e^{-2\ln n} = \frac{2}{n^2}$$

using union bound and summing over all bins:

$$\Pr\left[\text{any bin load} \geq \frac{3\ln n}{\ln\ln n}\right] \leq \frac{2}{n}$$

Exercise: It can be shown that maximum load is at least $\Omega\left(\frac{\ln n}{\ln\ln n}\right)$.

if we used Markov's inequality:

- let $b_i = 1/0$ iff ball i lands in a fixed bin
- L : load on a fixed bin

$$E[L] = \sum_{i=1}^{n} E[b_i] = \sum_{i=1}^{n} \frac{1}{n} = 1$$

- then $\Pr[L \geq k] = \Pr\left[\mathcal{E}_i^{\geq k}\right] \leq \frac{1}{k}$

  this is not strong enough to bound maximum load.

If we used Chebyshev's:

$$\mathrm{Var}[L] = \mathrm{Var}\left[\sum_{i=1}^{n} b_i\right] = \sum_{i=1}^{n} \mathrm{Var}[b_i] \leq \sum_{i=1}^{n} E[b_i] = 1$$

$$\Pr[L \geq k] \leq \Pr[L \geq t] \leq \frac{\mathrm{Var}[L]}{t^2} = \frac{1}{t^2}$$

$$\Pr\left[L \geq \sqrt{n\ln n}\right] \leq \frac{1}{n\ln n}$$

$\rightarrow$ Probability of maximum load $\geq \sqrt{n\ln n}$ is $\leq \frac{1}{\ln n}$.

if we used Chernoff's bound:

$$\Pr[|L - E[L]| \geq t] < 2^{-t}$$

with $t = 2\log n$ $\Pr[L \geq 2\log n] \leq \frac{1}{n^2}$

So with probability of max load $\leq 2\log n$ is at least $1 - \frac{1}{n}$.

Note: This is still weaker than $O(\frac{\ln n}{\ln\ln n})$ we obtained. Using stronger version of Chernoff we can obtain the same stronger bound.

Martingles (Azuma inequality)

- Back to Question 2: expected # of empty bins is n/e.

  Can we use Chernoff bound to show this holds w.h.p.?

- Note: events are not independent!

  Conditional on the event that bin i is empty the probability of other bins being empty decreases!

Azuma's inequality: Let X be a random variable

that is determined by n trials $T_1, \ldots, T_n$, s.t.
for each i and any two sequence of possible
outcomes $t_1, t_2, \ldots, t_{i-1}, t_i$ and $t_1, t_2, \ldots, t_{i-1}, t_i'$

$$|E[X \mid T_1 = t_1, \ldots, T_i = t_i] - E[X \mid T_1 = t_1, \ldots, T_i = t_i']| \leq c_i$$

i.e. changing the outcome of trial i while others are fixed, does not change $E[X]$ by more than c_i.

Then $$\Pr[|X - E[X]| > t] \leq 2e^{-t^2/2\sum_{i=1}^{n} c_i^2}.$$

- Azuma's inequality is one the most powerful.
- In the case of Balls & Bins, changing the outcome of one trial does not change the # of empty bins by more than 1; So $c_i = 1$.

$$\Pr[|X - \mu| > \sqrt{n \ln n}] \leq 2e^{\frac{-n \ln n}{2n}} \leq \frac{2}{n}$$

- Thus w.h.p., the number of empty bins X:

$$X \sim \frac{n}{e} \pm \Theta(\sqrt{n \ln n})$$

example: Consider a gambling setting in which you can bet any amount up to B for each round.

Assume it is a fair game (prob. of win/lose is the same)
if you win you gain the amount you bet (same for loss); so a coin flip win/lose.

if X_t is the result of coin flip in round t and we define Y_t as the total gain after t rounds then $|Y_t - Y_{t-1}| \leq B$ and expected value of gain in each round is zero.

So (regardless of our strategy and how much we bet in each round), using Azuma inequality:

$$\Pr[Y_t \geq \lambda] \leq 2e^{-\lambda^2/2tB^2}$$

Power of two choices [Mitzenmacher'91]

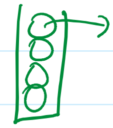
- For the Balls & Bins suppose for each ball we select two bins u.r. (instead of one) and the ball is placed in the bin with the lower load.
- Question: what is the maximum load among bins?

Goal: Show this reduces the max load to $O(\ln \ln n)$

Definition: Ball b has height i if it is i'th ball

$|O| \to$ height $= 4$

Definition: Ball b has height i if it is i'th ball thrown in that bin;



height/load of a bin $\equiv$ max height of balls in it

Some intuition:

- h_i : the event a ball gets height i
- # of bins of height ≥ 4 is $\leq \frac{n}{4}$.
- The probability of ball i gets height ≥ 5: both bins selected have ≥ 4 balls

$$\Pr[h_5] \leq \left(\frac{1}{4}\right)^2 \longrightarrow$$

$$E[\text{\# of bins w. height} \geq 5] \leq n\left(\frac{1}{4}\right)^2 = \frac{n}{4^2}$$

- Similarly: $\Pr[h_6] \leq \left(\frac{1}{4^2}\right)^2$

  and generally $\Pr[h_{i+1}] \leq \left(\frac{1}{4}\right)^{2^{i-1}}$

  for $i \sim \log\log n$: $\left(\frac{1}{4}\right)^{2^i} \approx \frac{1}{n}$.
- These are under the assumption values are always close to expectation.
- Binomial random variable X, denoted $B(n,p)$: is flipping n coins with head probability p

$$\Pr[X=i] = \begin{cases} \binom{n}{i} p^i (1-p)^{n-i} & i=0,1,2,\dots \end{cases}$$

$$\Pr[X=i] = \begin{cases} \binom{n}{i} p^i (1-p)^{n-i} & i=0,1,2,\dots \\ 0 & \text{o.w.} \end{cases}$$

- We use the following two lemmas.

Lemma 1: $\Pr[B(n,p) > 2np] < e^{-np/3}$

Proof: Use Chernoff bound!

Lemma 2: Suppose $x_1 \dots x_n$ and $y_1 \dots y_n$ are random variables s.t. $y_i = y_i(x_1, \dots, x_n)$ with $\Pr[y_i = 1 \mid x_1 \dots x_n] \le p$.

Then $\Pr[\sum_i y_i > k] \le \Pr[B(n,p) > k]$.

Definition: $\nu_i(t)$: # of bins of height $\ge i$ after t

$\mu_i(t)$: # of balls of height $\ge i$ after time t

h_t: height of ball t.

Note: $\mu_i(t) \ge \nu_i(t)$

Def: $\mathcal{E}_i$: event $\nu_i(t) \le \beta_i n$ where

$$\beta_4 = \frac{1}{4}, \quad \beta_{i+1} = \beta_i^2 \qquad i \ge 4$$

Lemma 3: $\forall\, i \ge 4$, $\mathcal{E}_i$ holds.

Proof: Induction.

Base: $i=4$: $\nu_4(t) \le \frac{n}{4}$, so $\mathcal{E}_4$ holds

Induction step: assum $\mathcal{E}_i$ holds $(i \geq 4)$:

$$\nu_i(t) \leq \beta_i n.$$

$$Y_t = 1 \quad \underline{\text{iff}} \quad (h_t \geq i+1 \text{ and } \nu_i(t-1) \leq \beta_i n)$$

Note: $\begin{cases} \text{if } \nu_i(t-1) \leq \beta_i n \longrightarrow \Pr[Y_t = 1] \leq \beta_i^2 \\ \text{else} \qquad \Pr[Y_t = 1] = 0 \end{cases}$

$\longrightarrow$ $\Pr[Y_t = 1] \leq \beta_i^2$, So Y_t is dominated by $B(n, \beta_i^2)$

$$\longrightarrow \Pr\left[\sum_{t=4}^{n} Y_t > 2\beta_i^2 n\right] \leq e^{-\beta_i^2 n/3} = e^{-\beta_{i+1} n/6}$$

Assume $\beta_{i+1} \geq \frac{12 \ln n}{n}$

Using the fact that $\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]} \leq \frac{\Pr[A]}{\Pr[B]}$

$$\Pr[\sim\mathcal{E}_{i+1} \mid \mathcal{E}_i] \leq \Pr\left[\sum_{t=4}^{n} Y_t \geq \beta_{i+1} n \,\middle|\, \mathcal{E}_i\right]$$

$$\leq \frac{1/n^2}{\Pr(\mathcal{E}_i)}$$

$$\longrightarrow \Pr[\sim\mathcal{E}_{i+1}] = \Pr[\sim\mathcal{E}_{i+1} \mid \mathcal{E}_i]\Pr[\mathcal{E}_i] + \Pr[\sim\mathcal{E}_{i+1} \mid \sim\mathcal{E}_i]\Pr[\sim\mathcal{E}_i]$$

$$\leq \frac{1}{n^2} + \Pr(\sim\mathcal{E}_i).$$

– Since $\Pr(\sim\mathcal{E}_4) = 0$, $\Pr(\sim\mathcal{E}_i)$ increases by $\leq \frac{1}{n^2}$ with each step i.

$B_4 = \frac{1}{4}$ $\quad B_5 = \left(\frac{1}{4}\right)^2$ $\quad B_6 = \left(\frac{1}{4^2}\right)^2$

Easy to check that for $i \leq \ln\ln n$: $\beta_i n \geq 12 \ln n$

- Easy to check that for $i \approx \log\log n$: $\beta_{i+1} n \geq \frac{12 \ln n}{n}$

- So things are good up to $i \approx \log\log n$.

- Let's see what happens when $\beta_{i+1} n < \frac{12 \ln n}{n}$

- let i^* be the first time $\beta_{i^*} < \frac{12 \ln n}{n}$.

Then

$$\Pr[\text{any fixed ball enters a bin with} \geq i^* \text{ balls}] \leq O\left(\frac{\ln^2 n}{n^2}\right)$$

$$\to \Pr[\text{any two balls enter a bin with} \geq i^* \text{ balls}]$$

$$\leq \binom{n}{2} O\left(\frac{\ln^4 n}{n^4}\right) \leq O\left(\frac{\ln^4 n}{n^2}\right)$$

$$\to \Pr[\text{any bin has} > i^* + 2 \text{ balls}] \leq O\left(\frac{\ln^4 n}{n^2}\right).$$

Extension: Huge improvement to go from 1 to 2 bins ($\Theta(\ln n / \ln\ln n)$ load to $\Theta(\ln\ln n)$).

Question: What if we select $\underline{d}$ bins and place the ball in the least loaded one?

(Azar, Broder, Karlin, Upfal '99): For any $d \geq 2$, the d-choice process gives a max-load of

$$\frac{\ln\ln n}{\ln d} \pm O(1) \quad \text{with probability } 1 - O\left(\tfrac{1}{n}\right)$$