Lecture 5 (Sept. 16): Undirected Cuts and Gomory-Hu Trees

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We discuss some properties of cuts in an undirected graph, and work toward finding Gomory-Hu trees which are a compact way to represent minimum $s-t$ cuts for all pairs of vertices s, t.

5.1 Cuts in Undirected Graphs

Let $\mathcal{G} = (V, E)$ be an undirected graph with capacities $\mu : E \to \mathbb{R}_{\geq 0}$. For $U \subseteq V$ let $\delta(U)$ denote all edges with precisely one endpoint in U.

Definition 1 *A* **cut** *in G is a subset $\emptyset \subsetneq U \subsetneq V$. An s–t* **cut** *for distinct $s, t \in V$ is a cut U with $|U \cap \{s, t\}| = 1$. We call U and $V - U$ the* **sides** *of the cut.*

Our definition of an $s-t$ cut in an undirected graph differs slightly from directed graphs in that it does not matter which side of the cut contains s. Note that $\delta(U) = \delta(V - U)$ for each $U \subseteq V$ in an undirected graph.

Definition 2 *The* Global Minimum Cut *problem is to find a cut U with minimum capacity $\mu(\delta(U))$.*

Since a cut contains at least one vertex and excludes at least one vertex, we can compute a global minimum cut in polynomial time by trying all distinct pairs of vertices (s, t) and computing the minimum $s-t$ cut in $\mathcal{G}$. Output the cheapest solution found. This requires $O(n^2)$ calls to a maximum flow calculation.

We can rely on fewer maximum flow calculations. Pick an arbitrary vertex s. Now, s lies on some side of the cut and we try all $n-1$ guesses for a vertex t separated from s in the global minimum cut. Computing the $n-1$ different minimum $s-t$ cuts for various t uses only $O(n)$ maximum flow calculations.

This idea almost works in directed graphs. Fix some s and guess all $n-1$ vertices t separated from s in a minimum cut. The main difference is that we have to compute a minimum $s-t$ cut and a minimum $t-s$ cut, which uses a total of $2n-2$ maximum flow calculations.

We can do much more with $O(n)$ maximum flow calculations. We will describe and construct an appropriate data structure that compactly represents all minimum $s-t$ cuts for all pairs $s, t \in V$. This lecture defined Gomory-Hu trees and established helpful properties about such trees and, in general, about cuts in an undirected graph. Next lecture presents the algorithm.

5.2 Gomory-Hu Trees

Throughout we will fix an **undirected** graph $\mathcal{G} = (V, E)$ with capacities $\mu : E \to \mathbb{R}_{\geq 0}$.

Definition 3 *For $u, v \in V$ let $\lambda_{u,v}$ be the capacity of a minimum $u-v$ cut.*

There is a neat relationship between these values.

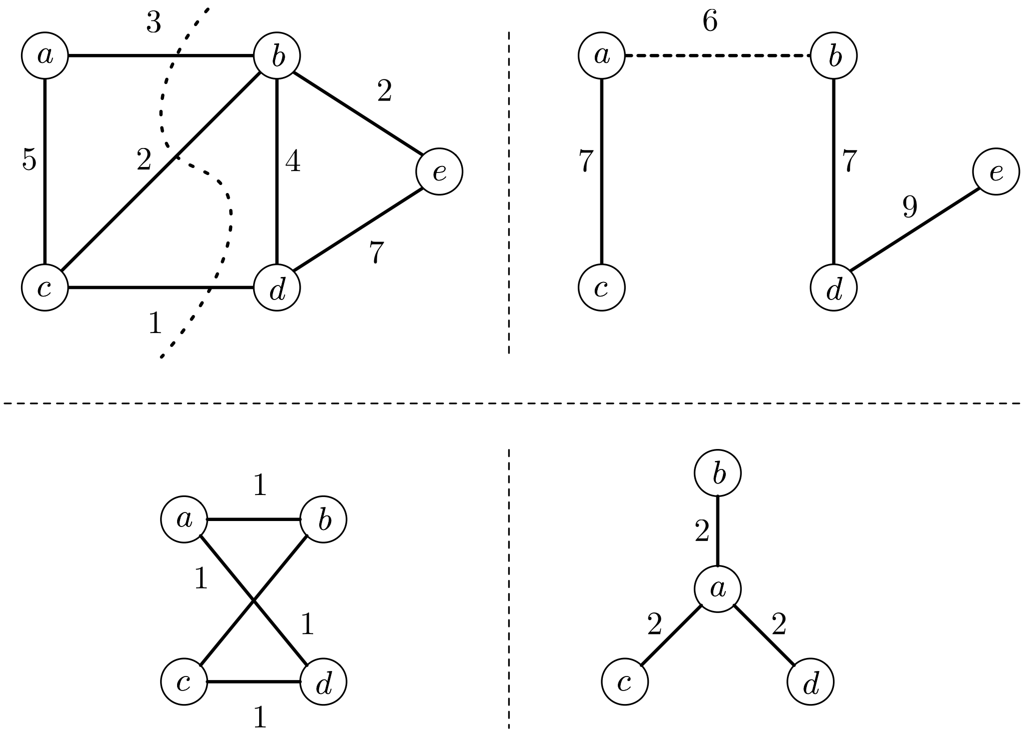

Figure 5.1: Two examples. The upper-right tree is a GHT for the upper-left graph. The edge weights in the tree denote the capacity of the corresponding fundamental cuts. The dashed edge in the tree corresponds to the fundamental cut $\{a, b\}$, which is highlighted on the upper-left graph with the dashed curve. The lower example shows that it may be $F \not\subseteq E$.

Lemma 1 *Let $v_1, v_2, \ldots, v_k$ be any sequence of $k \geq 2$ distinct vertices. Then $\lambda_{v_1,v_k} \geq \min_{1 \leq i \leq k-1} \lambda_{v_i,v_{i+1}}$.*

Proof. Let U be a minimum $v_1 - v_k$ cut and suppose, by replacing U with $V - U$ instead if necessary, that $v_1 \in U, v_k \notin U$. Then there is some $1 \leq i \leq k-1$ with $v_i \in U, v_{i+1} \notin U$. This U is also a $v_i - v_{i+1}$ cut so $\lambda_{v_i,v_{i+1}} \leq \mu(\delta(U)) = \lambda_{v_1,v_k}$. ■

Definition 4 *Let $T = (W, F)$ be a tree and $e \in F$. Then set of nodes C_e of a connected component of $(W, F - \{e\})$ is a* **fundamental cut** *for e.*

There are two fundamental cuts for each edge uv of a tree, one containing u the other containing v. We often speak of **the** fundamental cut of an edge when it doesn't matter which side we pick.

Definition 5 *A* **Gomory-Hu Tree** *(***GHT***) for $(\mathcal{G}, \mu)$ is a tree $T = (V, F)$ on the same set of vertices as $\mathcal{G}$ such that for every $uv \in F$, the fundamental cut C_{uv} is a minimum $u - v$ cut in G (i.e. $\mu(\delta(C_{uv})) = \lambda_{u,v}$).*

Two examples are shown in Figure 5.1. Note that it is not necessarily true that $F \subseteq E$, as in the second example in the figure.

It is not clear that a graph necessarily has an associated GHT. We will show this is indeed the case and we can compute it using $O(n)$ calls to a maximum flow algorithm, plus some relatively simple processing.

While the definition seems to only give information about minimum cuts for every pair $u, v \in V$ that appear as an edge of T, the following shows how to easily find a minimum u, v cut for every pair of nodes.

Lemma 2 *Let $T = (V, W)$ be a GHT for $\mathcal{G}$. For any distinct $u, v \in V$ let P^{uv} denote the edges on the unique path between u and v on T. Let $ab \in P^{uv}$ achieve $\min_{ab \in P^{uv}} \lambda_{a,b}$. Then $\lambda_{u,v} = \lambda_{a,b}$ and the fundamental cut C_{ab} is also a minimum $u - v$ cut.*

Proof. On one hand, Lemma 1 shows $\lambda_{u,v} \geq \lambda_{a,b}$ by considering the sequence of nodes on the $u - v$ path in T.

On the other hand, $\lambda_{u,v} \leq \lambda_{a,b}$ because the fundamental cut C_{ab} is also a $u - v$ cut and (by definition of a GHT) has capacity $\lambda_{a,b}$.

Putting these two bounds togeter, we see $\lambda_{u,v} = \lambda_{a,b}$. Also, C_{ab} is a minimum $u - v$ cut because it is a $u - v$ cut with capacity $\lambda_{a,b}$. ■

5.2.1 Submodularity of Cuts

To motivate the algorithm that finds (and demonstrates existence of) a GHT, we further explore properties of cuts.

Theorem 1 (Submodularity of Cuts) *For $A, B \subseteq V$ we have $\mu(\delta(A)) + \mu(\delta(B)) \geq \mu(\delta(A \cap B)) + \mu(\delta(A \cup B))$.*

Proof. We count how many times each $e \in E$ contributes to both sides of the claimed bound. So consider some $e = uv$.

- If $uv \in \delta(A \cap B)$ with, say, $u \in A \cap B$ then either $v \notin A$ or $v \notin B$. This means either $uv \in \delta(A)$ or $uv \in \delta(B)$.
- If $uv \in \delta(A \cup B)$ with, say, $v \notin A \cup B$ then either $u \in A$ or $u \in B$. This means either $uv \in \delta(A)$ or $uv \in \delta(B)$.
- Finally, if $uv \in \delta(A \cap B)$ **and** $uv \in \delta(A \cup B)$ with, say, $u \in A \cap B$ then $v \notin A$ and $v \notin B$ so $uv \in \delta(A)$ **and**$uv \in \delta(B)$.

■

This allows us to prove a sort of nested property of some cuts. This will be exploited heavily in the construction of a Gomory-Hu tree.

Lemma 3 *Let $s, t \in V$ be distinct vertices and A a minimum $s - t$ cut. Let $u, v \notin A$ be distinct vertices (it could be that either u or v equals either s or t). Then there is a minimum $u - v$ cut B with $A \subseteq B$ or $A \cap B = \emptyset$.*

Note the statement is slightly redundant: if we have some minimum $u - v$ cut B has $A \subseteq B$ then $V - B$ is a minimum $u - v$ cut with $A \cap B = \emptyset$ and vice-versa.

Proof. Let B be some minimum $u - v$ cut. We suppose $A \not\subseteq B$ and $A \cap B \neq \emptyset$, otherwise we are done. We will show how to modify B to get another minimum $u - v$ cut with the desired properties.

Without loss of generality (by renaming if necessary), we suppose $s \in A$. If $s \notin B$ then we replace B with $V - B$. Finally, again by renaming u and v if necessary we assume $u \in B$.

By submodularity of cuts we have

$$\mu(\delta(A)) + \mu(\delta(B)) \geq \mu(\delta(A \cap B)) + \mu(\delta(A \cup B)). \tag{5.1}$$

By assumption, $s \in A \cap B$ and $t \notin A \cap B$ so the fact that A is a minimum $s - t$ cut means $\mu(\delta(A)) \leq \mu(\delta(A \cap B))$. By this and (5.1),

$$\mu(\delta(A \cup B)) \leq \mu(\delta(B)). \tag{5.2}$$

Now, $u \in A \cup B$ and $v \notin A \cup B$ so $A \cup B$ is a $u - v$ cut. Then (5.2) shows $A \cup B$ is a minimum $u - v$ cut which clearly contains A. ■