CMPUT 675: Topics in Combinatorics and Optimization **Fall 2016**

Lecture 33-34 (Nov 30 - Dec 2): Perfect Graphs

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One of our last topics concerns perfect graphs. These are a fairly general class of graphs with a nice polyhedral characterization of the convex hull of independent sets. We can find maximum independent sets and minimum colourings in such graphs in polynomial time, but to date the only polynomial-time algorithms that do so utilize semidefinite programming!

To further motivate our discussion, we take a polyhedral perspective. For a graph $\mathcal{G} = (V; E)$, consider the following "convex relaxation" of the set of independent sets of $\mathcal{G}$:

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^V_{\geq 0} : \mathbf{x}_u + \mathbf{x}_v \leq 1 \text{ for each } uv \in E\}.$$

Earlier we saw $\mathcal{P}$ has integral extreme points if and only if $\mathcal{G}$ is bipartite. This is a fairly weak relaxation in general graphs. Setting $\mathbf{x}_v = \frac{1}{2}$ for each $v \in V$ is feasible even if $\mathcal{G}$ is a large clique!

We consider a stronger polytope now, based on the observation that a clique can contain at most one vertex from any independent set.

$$\mathcal{P}_{\text{clique}} = \{\mathbf{x} \in \mathbb{R}^V_{\geq 0} : \mathbf{x}(C) \leq 1 \text{ for each clique } C\}.$$

This is also a somewhat weak relaxation: there are graphs that do not contain any size-3 cliques that still have very small independent sets, so setting $\mathbf{x}_v = \frac{1}{2}$ is still feasible and greatly overestimates the size of the largest independent set.

But for other graphs, this relaxation is much stronger. Our motivation is to study the class of graphs for which $\mathcal{P}_{\text{clique}}$ is integral (i.e. has integral extreme points).

33.1 Perfect Graphs

The following notation will be used throughout the lectures. Let $\mathcal{G} = (V; E)$ be an undirected graph.

- $\alpha(\mathcal{G})$ is the size of a maximum **independent set** in $\mathcal{G}$.
- $\omega(\mathcal{G})$ is the size of a maximum **clique** in $\mathcal{G}$.
- $k(\mathcal{G})$ the size of a minimum **clique cover**. This is the value ℓ such that V can be partitioned into ℓ cliques.
- $\chi(\mathcal{G})$ the **chromatic number** of $\mathcal{G}$. This is the minimum number of colours we can use to colour vertices such that no edge has endpoints with the same colour.

Note, in general graphs, these are all **NP**-hard to compute.

Finally, let $\overline{\mathcal{G}} = (V; \overline{E})$ be the graph obtained from $\mathcal{G}$ by "complementing" the edge set. That is, $\overline{E} = \{uv : u, v \in V, u \neq v, uv \notin E\}$.

Lemma 1 *For any graph $\mathcal{G}$, $\alpha(\mathcal{G}) \leq k(\mathcal{G})$ and $\omega(\mathcal{G}) \leq \chi(\mathcal{G})$.*

Proof. Let U be a maximum independent set. Any clique C has $|C \cap U| \leq 1$ so there are at least $|U|$ cliques in a clique covering of $\mathcal{G}$. Note $\omega(\mathcal{G}) = \alpha(\overline{\mathcal{G}})$ (cliques in $\mathcal{G}$ are independent sets in $\overline{\mathcal{G}}$) and $\chi(\mathcal{G}) = k(\overline{\mathcal{G}})$ (a partition into independent sets in $\mathcal{G}$ is a partition into cliques in $\overline{\mathcal{G}}$), so

$$\omega(\mathcal{G}) = \alpha(\overline{\mathcal{G}}) \leq k(\overline{\mathcal{G}}) = \chi(\mathcal{G}).$$

■

Finally, for $U \subseteq V$ we let $\mathcal{G}[U] = (U; E[U])$ where $E[U] = \{uv \in E : u, v \in U\}$. This is the **subgraph of $\mathcal{G}$ induced by U**.

Definition 1 *A graph $\mathcal{G}$ is perfect if and only if $\omega(\mathcal{G}[U]) = \chi(\mathcal{G}[U])$ for each $U \subseteq V$.*

Simply calling $\mathcal{G}$ perfect if $\omega(\mathcal{G}) = \chi(\mathcal{G})$ is not nearly as interesting. For example, we could add a disjoint clique of size $\chi(\mathcal{G})$ to $\mathcal{G}$ to get a new graph whose largest clique equals its chromatic number.

So we stick with this more robust definition. This will be a far more interesting subject to study.

One of two goals for today is to prove the following **perfect graph theorem**, sometimes known as the **weak perfect graph theorem** (in deference to the strong perfect graph theorem stated below).

Theorem 1 (Lovász (1972)) *$\mathcal{G}$ is perfect if and only if $\overline{\mathcal{G}}$ is perfect.*

If so, then we can equivalently call a graph perfect if $\alpha(\mathcal{G}[U]) = k(\mathcal{G}[U])$ for each $U \subseteq V$.

Before proving this, here are some examples.

Examples:

- **Bipartite graphs**: If they have an edge, the maximum clique size is 2 and they are, by definition, 2-colourable.
- **Line graphs of bipartite graphs**: Let $\mathcal{G} = (V; E)$ be a graph. The line graph $\mathcal{L}(\mathcal{G})$ has vertices E where two are adjacent if and only if the corresponding edges in $\mathcal{G}$ share an endpoint.

  Vizing's theorem (which we never visited, but is simple to follow) says $\chi(\mathcal{L}(G))$ is the maximum degree of a vertex in $\mathcal{G}$ if $\mathcal{G}$ is bipartite. The maximum degree in $\mathcal{G}$ is the maximum clique size in $\mathcal{L}(\mathcal{G})$ for any graph (unless all vertices have degree ≤ 2 and the graph contains a triangle, but that won't happen in bipartite graphs). So $\mathcal{L}(\mathcal{G})$ is perfect if $\mathcal{G}$ is bipartite.
- **Comparability graphs**: These are undirected graphs obtained from acyclic and transitive (i.e $uv, vw \in E$ means $uw \in E$) directed graphs by ignoring the directions. In the first assignment, you prove the maximum antichain equals the minimum chain cover. In the underlying undirected graph, this means the maximum independent set size equals the minimum clique cover size.
- **Chordal graphs**: Graphs such that any cycle of length at least 4 has a **chord**. I.e. if $v_1, v_2, \ldots, v_k$ is a cycle for some $k \geq 4$ then there is an edge of the form $v_i v_j$ where $v_i v_j$ are not adjacent vertices on the cycle.

  Unfortunately we did not have time to discuss this beautiful class of graphs! For example, they are exactly the graphs that have the following representation: there is a tree $\mathcal{T}$, each $v \in V$ (from $\mathcal{G}$) corresponds to a connected subtree of $\mathcal{T}$ where u, v are adjacent in $\mathcal{G}$ if and only if their corresponding subtrees share a vertex in common.

  Note, as a special case this includes the interval graphs you saw on Assignment 3.

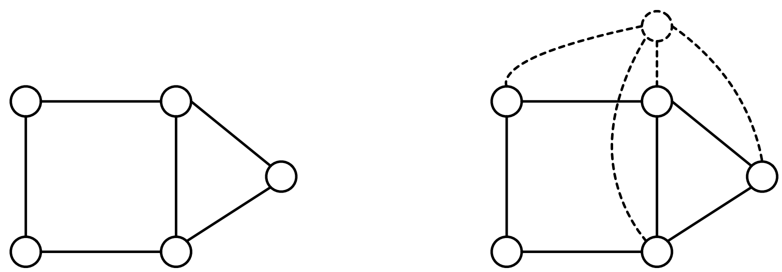

Figure 33.1: Replication of a vertex.

There is an alternative characterization of perfect graphs we cannot prove in this class.

The class of graphs $\mathcal{G}$ such that neither $\mathcal{G}$ nor $\overline{\mathcal{G}}$ contain a cycle of odd length $k \geq 5$ as an induced subgraph are called **Berge** graphs, named after Claude Berge who conjectured in 1961 they were exactly the class of perfect graphs. It took quite a while, but it was eventually proven to be true.

Strong Perfect Graph Theorem

Theorem 2 (Chudnovsky, Seymour, Thomas, and Robin (2006)) *A graph $\mathcal{G}$ is perfect if and only if it is a Berge graph.*

One direction is clear, an odd-length cycle C_ℓ with $\ell \geq 5$ is not perfect as $k(C_\ell) = 2$ yet $\chi(C_\ell) = 3$.

Note, also, that the strong perfect graph theorem easily implies the weak perfect graph theorem, as clearly $\mathcal{G}$ is Berge if and only if $\overline{\mathcal{G}}$ is Berge.

33.2 The Perfect Graph Theorem

Let $\mathcal{G} = (V; E)$ be a graph and let $v \in V$. Let $\mathcal{G}^{+v}$ denote the following graph. The vertex set is $V \cup \{v'\}$ where v' is a new vertex (think of it as a *copy* of v). Each edge of E is an edge in the new graph. Further, v' is adjacent to v and to every other vertex in $\delta_E(v)$. See Figure 33.1.

Lemma 2 (Replication Lemma) *For any perfect graph $\mathcal{G} = (V; E)$ and any $v \in V$, $\mathcal{G}^{+v}$ is perfect.*

Proof. We prove this by induction on $|V|$. The case $|V| = 1$ is trivial to check.

We can colour $\mathcal{G}^{+v}$ with $\alpha(\mathcal{G})+1$ colours by first colouring $\mathcal{G}$ and then using a new colour for v'. If v is contained in a maximum clique C of $\mathcal{G}$ then $C \cup \{v'\}$ is a clique of size $\omega(\mathcal{G}^{+v}) = \omega(\mathcal{G}) + 1$. In this case, $\alpha(\mathcal{G}^{+v}) = \omega(\mathcal{G}^{+v})$, so we assume v is not contained in any maximum clique of $\mathcal{G}$.

Note this means $\omega(\mathcal{G}^{+v}) = \omega(\mathcal{G})$: if $|C'| = |C| + 1$ where C' is a maximum clique in $\mathcal{G}^{+v}$ and C is a maximum clique in $\mathcal{G}$ then $v' \in C'$, so $v \in C'$ as well (otherwise $C' \cup \{v\}$ would be an even larger clique). But then $C' - v$ would be a maximum clique in $\mathcal{G}$ containing v, contradicting our assumption.

Now let $\mathcal{G}' = \mathcal{G}[V - \{v\}]$. By assumpion, $\omega(\mathcal{G}') = \omega(\mathcal{G})$. Let ϕ be a colouring of $\mathcal{G}$ with $\chi(\mathcal{G}) = \omega(\mathcal{G})$ colours (recall $\mathcal{G}'$ is an induced subgraph of $\mathcal{G}$). Let $X = \{u \in V : \phi(u) = \phi(v)\}$ and observe X is an independent set.

Observe $\chi(\mathcal{G}[V - X]) = \chi(\mathcal{G}) - 1$, as we can extend a colouring of $\mathcal{G}[V - X]$ to one for $\mathcal{G}$ by using a new colour for X. Because $\mathcal{G}[V - X]$ is an induced subgraph of the perfect graph $\mathcal{G}$,

$$\omega(\mathcal{G}[V - X]) = \omega(G) - 1.$$

We assumed v does not lie in a maximum clique of $\mathcal{G}$, so removing only $X - v$ suffices to drop the clique number:

$$\omega(\mathcal{G}[V - (X - \{v\})]) = \omega(G) - 1.$$

Finally, $X - \{v\} \cup \{v'\}$ is also an independent set in $\mathcal{G}^{+v}$, so if we remove it from $\mathcal{G}^{+v}$ we get $\mathcal{G}[V - (X - \{v\})]$ which can be coloured with $\omega(G) - 1$ colours, thus extending to a colouring of $\mathcal{G}^{+v}$ with $\omega(G)$ colours. That is,

$$\chi(\mathcal{G}^{+v}) = \omega(\mathcal{G}) = \omega(\mathcal{G}^{+v}).$$

■

The next step of the proof of the weak perfect graph theorem requires a classification of the **stable set** polytope of perfect graphs. This was part of our motivation for studying perfect graphs anyway.

Theorem 3 *If $\mathcal{G}$ is perfect then every extreme point of $\mathcal{P}_{\text{clique}} := \{\mathbf{x} \in \mathbb{R}^V_{\geq 0} : \mathbf{x}(C) \leq 1$ for each clique $C \subseteq V\}$ is integral.*

Proof. Let $\mathbf{x}$ be an extreme point. Because $\mathbf{x}$ is rational, there is some $\eta \in \mathbb{Z}_{\geq q}$ such that $\eta \cdot \mathbf{x}_v \in \mathbb{Z}_{\geq 0}$ for each $v \in V$. We will show $\mathbf{x}$ can be written as a convex combination of integer points in $\mathcal{P}_{\text{clique}}$. By what you showed in Exercise 3, this means $\mathcal{P}_{\text{clique}}$ is integral.

Consider the graph perfect graph $\mathcal{G}'$ obtained from $\mathcal{G}$ in the following way. First delete every vertex v with $\eta \cdot \mathbf{x}_v = 0$ and call these deleted vertices U. Then apply the construction described before Lemma 2 to each vertex $v \in V - U$ so that $\eta \cdot \mathbf{x}_v$ copies (including the original) are present.

In other words, by Lemma 2 $\mathcal{G}' = (V'; E')$ is perfect where $V' = \{(v, i) : v \in V - U, 1 \leq i \leq \eta \cdot \mathbf{x}_v\}$ and $E' = \{(u, i)(v, j) : u = v \text{ or } uv \in E\}$.

Let C' be a maximum clique in $\mathcal{G}'$.The set of all $v \in V$ appearing with some index in C' is a clique C in $\mathcal{G}$. So by feasibility of $\mathbf{x}$,

$$\omega(\mathcal{G}') = |C'| = \sum_{v \in C} |\{(v, i) \in C' : 1 \leq i \leq \eta \cdot \mathbf{x}_v\}| \leq \sum_{v \in C} \eta \cdot \mathbf{x}_v \leq \eta.$$

Colour $\mathcal{G}'$ with η colours. That is, we know $\omega(\mathcal{G}') \leq \eta$ colours suffice (because $\mathcal{G}'$ is perfect), so pad the colouring with "unused" colours if necessary and view it as using exactly η colours. For each colour $\kappa = 1, \ldots, \eta$ let $\mathbf{a}_\kappa \in \mathbb{R}^V$ be such that $\mathbf{a}_\kappa(v) = 1$ if some copy of v in $\mathcal{G}'$ is coloured j and 0 otherwise.

Note

$$\frac{1}{\eta} \sum_{j=1}^{\eta} \mathbf{a}_j = \mathbf{x}.$$

That is, $\mathbf{x}$ can be represented as a convex combination of integral points in $\mathcal{P}_{\text{clique}}$.

Since $\mathbf{x}$ is already an extreme point, then this convex combination must be trivial (i.e. all $\mathbf{a}_j$ are the same) so $\mathbf{x}$ is integral. ■

Unfortunately, we cannot separate the exponentially many constraints defining $\mathcal{P}_{\text{clique}}$ in polynomial time for an arbitrary graph. Otherwise, we could determine the size of a maximum clique by finding the largest integer b such that $\mathbf{x}_v = \frac{1}{b}$ is feasible.

Later we will see how to optimize over this LP for perfect graphs and, more generally, how to find a feasible solution $\mathbf{x}$ for this LP in arbitrary graphs that satisfies the constraint and is at least as valuable as the optimum integer solution. Surprisingly, this will use semidefinite programming!

We now return to the proof of the perfect graph theorem. The proof will follow directly from the following result and Theorem 3. Chaining these two results shows $\mathcal{G}$ is perfect implies $\overline{\mathcal{G}}$ is perfect. Of course, applying this sequence again to $\overline{\mathcal{G}}$ shows the other direction. It also shows $\mathcal{G}$ is perfect if and only if $\mathcal{P}_{\text{clique}}$ is perfect.

Theorem 4 *Let $\mathcal{G} = (V; E)$ be a graph. If $\mathcal{P}_{\text{clique}} := \{\mathbf{x} \in \mathbb{R}^V_{\geq 0} : \mathbf{x}(C) \leq 1$ for each clique $C \subseteq V\}$ is integral then $\overline{\mathcal{G}}$ is perfect.*

Proof. By induction on $|V|$. Any graph with $\mathcal{G}$ at most 2 vertices trivially has $\overline{\mathcal{G}}$ being perfect.

By our definition of perfect, we know $\chi(\mathcal{G}[U]) = \omega(\mathcal{G}[U])$ for every $U \subseteq V$. Our goal is to show $\alpha(\mathcal{G}[U]) = k(\mathcal{G}[U])$ for every $U \subseteq V$. By induction, it suffices to show $\alpha(\mathcal{G}) = k(\mathcal{G})$.

Consider $\mathcal{P}' := \mathcal{P}_{\text{clique}} \cap \{\mathbf{x} \in \mathbb{R}^V_{\geq 0} : \mathbf{x}(V) = \alpha(\mathcal{G})\}$. By integrality of $\mathcal{P}_{\text{clique}}$, every $\mathbf{x} \in \mathcal{P}_{\text{clique}}$ has $\mathbf{x}(V) \leq \alpha(\mathcal{G})$. Thus, $\mathcal{P}'$ is a **supporting hyperplane** of $\mathcal{P}_{\text{clique}}$ so at least one constraint defining $\mathcal{P}_{\text{clique}}$ is tight for all $x \in \mathcal{P}'$ (we did not prove this explicitly but it follows from general facts about supporting hyperplanes of polytopes, see Chapter 3.1 of the course textbook for discussion). This cannot be a nonnegativity constraint as $\mathbf{x}(V) = \alpha(\mathcal{G}) \geq 1$ for all $\mathbf{x} \in \mathcal{P}_{\text{clique}}$.

So there is some clique C such that $\mathbf{x}(C) = 1$ for all $\mathbf{x} \in \mathcal{P}'$. That is, C intersects *every* maximum independent set of $\mathcal{G}$. By induction, $\mathcal{G}[V - C]$ can be partitioned into $\alpha(\mathcal{G}[V - C]) = \alpha(\mathcal{G}) - 1$ cliques, so adding C to this clique partitioning yields a partitioning of $\mathcal{G}$ using $\alpha(\mathcal{G})$ cliques. ■

33.3 The Theta Body of a Graph

Even though separating the constraints of $\mathcal{P}_{\text{clique}}$ is **NP**-hard, we can still circumvent this difficulty using a trick. Namely, consider the following convex set associated with a graph. Suppose $V = \{1, \ldots, n\}$ and that 0 is a new *index* not in V. We let $V' := V \cup \{0\}$.

Definition 2 *The* **theta body** *of $\mathcal{G}$, denoted $TH(G)$, is all matrices $\mathbf{X} \in \mathbb{R}^{V' \times V'}$ satisfying the following constraints:*

$$\begin{array}{rcll} \mathbf{X}_{0,v} & = & \mathbf{X}_{v,v} & \text{for each } v \in V \\ \mathbf{X}_{u,v} & = & 0 & \text{for each } uv \in E \\ \mathbf{X}_{0,0} & = & 1 & \\ \mathbf{X} & \succeq & 0 & \end{array}$$

Recall saying $\mathbf{X} \succeq 0$ means $\mathbf{X}$ is symmetric.

Definition 3 *For an undirected graph $\mathcal{G}$, let $\theta(\mathcal{G}) = \max_{\mathbf{X} \in TH(\mathcal{G})} \sum_{v \in V} \mathbf{X}_{v,v}$.*

The value $\theta(\mathcal{G})$ is expressed as the optimum value of a semidefinite program. We only have to guarantee that its feasible solutions are bounded to show that we can compute $\theta(\mathcal{G})$ within arbitrary precision.

Lemma 3 *Let $\mathbf{X} \in TH(\mathcal{G})$. Then $-1 \leq \mathbf{X}_{u,v} \leq 1$ for any $u, v \in V'$.*

Proof. We are explicitly given $\mathbf{X}_{0,0} = 1$. We also know $\mathbf{X}_{v,v} \geq 0$ as the diagonal of any p.s.d. matrix must be nonzero.

For $v \in V$, let $\mathbf{z} \in \mathbb{R}^{V'}$ be such that $\mathbf{z}_0 = 1$ and $\mathbf{z}_v = -1$ for a value t. Then

$$0 \leq \mathbf{z}^T \cdot \mathbf{X} \cdot \mathbf{z} = \mathbf{X}_{0,0} - 2 \cdot \mathbf{X}_{0,v} + \mathbf{X}_{v,v} = 1 - \mathbf{X}_{v,v}.$$

Here we used symmetry of $\mathbf{X}$ and $\mathbf{X}_{0,v} = \mathbf{X}_{v,v}$. This shows $\mathbf{X}_{v,v} \leq 1$.

So we have a p.s.d. matrix with bounded trace. This means all entries must be bounded in value. Explicitly, for $u, v \in V'$ we let $\mathbf{z}_u = 1, \mathbf{z}_v = \pm 1$ and $\mathbf{z}_w = 0$ for $w \neq u, v$. Then

$$0 \leq \mathbf{z}^T \cdot \mathbf{X} \cdot \mathbf{z} = \mathbf{X}_{u,u} + \mathbf{X}_{v,v} \pm 2\mathbf{X}_{u,v}.$$

The two choices for ± 1 plus the fact the trace is bounded also bounds $\mathbf{X}_{u,v}$ between -1 and 1. ■

Now we will explore the relationship of $\theta(G)$ to other combinatorial values associated with a graph.

Theorem 5 (Sandwich Theorem) *For an undirected graph $\mathcal{G}$, $\alpha(\mathcal{G}) \leq \theta(\mathcal{G}) \leq k(\mathcal{G})$.*

Proof. Let $S \subseteq V$ be a maximum independent set. Let $y_v = 0$ for $v \notin S$ and $y_v = 1$ for $v \in S$. Also let $y_0 = 1$. Setting $\mathbf{X}_{u,v} = y_u \cdot y_v$ for $u, v \in V \cup \{0\}$ gives a point in $TH(\mathcal{G})$. That is, $\mathbf{X} \succeq 0$ because the values y_v, interpreted as 1-dimensional vectors, provide a Cholesky decomposition of $\mathbf{X}$. We have $\mathbf{X}_{u,v} = 0$ for $uv \in E$ because either y_u or y_v is 0 (as either $u \notin S$ or $v \notin S$). Finally, as $y_u^2 = y_u$ for each $u \in V$ we see $\mathbf{X}_{u,u} = y_u \cdot y_u = y_u = y_u \cdot y_0 = \mathbf{X}_{u,0}$. Note that $\sum_{v,v} \mathbf{X}_{v,v} = |S|$, so $\alpha(\mathcal{G}) \leq \theta(\mathcal{G})$.

For the other end, we first prove a claim.

Claim 1 *For any clique C, $\sum_{v \in C} \mathbf{X}_{v,v} \leq 1$.*

Proof. First we give some geometric intuition. Our final proof is algebraic, but this illustrates what is happening in $\mathbb{R}^{V \cup \{0\}}$. Let $\mathbf{x}_u \in \mathbb{R}^{V \cup \{0\}}$ be vectors for $u \in V \cup \{0\}$ such that $\mathbf{x}_u \circ \mathbf{x}_v = \mathbf{X}_{u,v}$ (as $\mathbf{X} \succeq 0$). Because $\mathbf{X} \in TH(\mathcal{G})$, the vectors $\{\mathbf{x}_u : u \in C\}$ are orthogonal to each other. But $\mathbf{X}_{v,v} = \mathbf{X}_{v,0} = \mathbf{x}_v \circ \mathbf{x}_0$ where $\mathbf{x}_0$ is a unit vector. So $\sum_{v \in C} \mathbf{X}_{v,v}$ is the sum of the squares of the lengths of the projections of the vectors $\{\mathbf{x}_v : v \in C\}$ onto the unit vector $\mathbf{x}_0$. This cannot be too long if the vectors are orthogonal (essentially this is Pythagoreas' theorem).

It is possible to convert this geometric approach into a rigorous proof using the identity $\sum_{\mathbf{v} \in B} (\mathbf{v} \circ \mathbf{u})^2 = \mathbf{u} \circ \mathbf{u}$ for any vector $\mathbf{u}$ and any orthonormal basis B of $\mathbb{R}^{V \cup \{0\}}$.

But our formal proof is algebraic. Simply put, let $\mathbf{z} \in \mathbb{R}^{V \cup \{0\}}$ be given by

$$\mathbf{z}_v = \begin{cases} 1 & \text{if } v = 0 \\ -1 & \text{if } v \in C \\ 0 & \text{if } v \in V - C \end{cases}$$

Then we see the following.

$$\begin{aligned} 0 &\leq \mathbf{z}^T \cdot \mathbf{X} \cdot \mathbf{z} \\ &= \mathbf{z}_0^2 \cdot \mathbf{X}_{0,0} + 2\mathbf{z}_0 \sum_{v \in V} \mathbf{z}_v \mathbf{X}_{v,0} + \sum_{u,v \in V} \mathbf{z}_u \mathbf{z}_v \mathbf{X}_{u,v} \\ &= 1 - 2 \sum_{v \in C} \mathbf{X}_{v,0} + \sum_{u,v \in C} \mathbf{X}_{u,v} \\ &= 1 - 2 \sum_{v \in C} \mathbf{X}_{v,v} + \sum_{v \in nC} \mathbf{x}_{v,v} \\ &= 1 - \sum_{v \in C} \mathbf{X}_{v,v} \end{aligned}$$

All steps are justified simply by definition of $\mathbf{z}$ and the fact that $\mathbf{X} \in TH(\mathcal{G})$. In particular, the second last equality holds because $\mathbf{X}_{u,v} = 0$ for each $uv \in E$ and any distinct $u, v \in C$ are connected by an edge (as C is a clique).

■

Returning the proof of the sandwich theorem, let $C_1, \ldots, C_{k(\mathcal{G})}$ be a minimum clique cover. By the claim above

$$\theta(\mathcal{G}) = \sum_{v \in C} \mathbf{X}_{v,v} = \sum_{i=1}^{k(\mathcal{G})} \sum_{v \in C_i} \mathbf{X}_{v,v} \leq \sum_{i=1}^{k(\mathcal{G})} 1 = k(\mathcal{G}).$$

■

The value $\theta(\mathcal{G})$, in general, is still not a good approximation for $\alpha(\mathcal{G})$ but it is exact when $\mathcal{G}$ is perfect!

Corollary 1 *There is a polynomial-time algorithm for computing $\alpha(G)$.*

Proof. We can compute $\theta(\mathcal{G})$ within arbitrary precision in polynomial time. Specifically, let $\epsilon = 1/3$ and use the Ellipsoid method to approximately solve the SDP that determines $\theta(\mathcal{G})$ within an additive error of $1/3$. Rounding the answer to the nearest integer then computes $\alpha(\mathcal{G})$ as $\alpha(\mathcal{G}) \leq \theta(\mathcal{G}) \leq k(\mathcal{G}) = \alpha(\mathcal{G})$ (so equality holds throughout) by the sandwich theorem and the fact $\mathcal{G}$ is perfect. ■

One can use an algorithm that computes $\alpha(\mathcal{G})$ into one that actually finds a maximum independent set.

Corollary 2 *We can compute an independent set S of size $\alpha(\mathcal{G})$ in polynomial time if $\mathcal{G}$ is perfect.*

Proof. If v lies in every independent set, then $\alpha(\mathcal{G} - v) = \alpha(\mathcal{G}) - 1$. Otherwise, $\alpha(\mathcal{G} - v) = \alpha(\mathcal{G})$. So while there is some vertex whose removal does not reduce the maximum independent set size, we remove it. Once this is done, the resulting graph $\mathcal{G}'$ has $\alpha(\mathcal{G}') = \alpha(\mathcal{G})$.

We claim $\mathcal{G}'$ has no edge, otherwise if there was an edge uv then either u or v could not be in every maximum independent set (as no independent set contains both u and v). ■

Corollary 3 *We can compute a clique C of size $\omega(\mathcal{G})$ in polynomial time if $\mathcal{G}$ is perfect.*

Proof. Compute a maximum independent set in $\overline{\mathcal{G}}$. ■

Colouring a perfect graph is also possible in polynomial time. But it takes a bit more work. It suffices to be able to compute maximum independent sets and cliques in polynomial time, so we do not need to devise new SDP techniques.

Theorem 6 *We can find a proper colouring of a perfect graph $\mathcal{G}$ in polynomial time using $\chi(\mathcal{G})$ colours.*

Proof. Recall we can compute the value $\chi(\mathcal{G})$ in polynomial time by computing $\alpha(\mathcal{G})$ (say, using Corollary 1). The issue is actually constructing such a colouring. To do this, it suffices to find an independent set S such that $\chi(\mathcal{G} - S) = \chi(\mathcal{G}) - 1$. If so, we use one colour for S and recursively colour $\mathcal{G} - S$ using $\chi(\mathcal{G}) - 1$ colours.

To find such a set S, we maintain a collection of maximum cliques $C_1, \ldots, C_\ell$, initially $\ell = 0$. We know there is an independent set S' that intersects all cliques: any colour class of a colouring with $\chi(\mathcal{G}) = \omega(\mathcal{G})$ colours will intersect each maximum clique.

To find such an S' that intersects each clique, compute a maximum-weight independent set S' where the weight w_v of $v \in V$ is the number of cliques C_i containing v. So independent sets with weight ℓ are precisely the independent sets that hit each clique C_i. This can also be done in polynomial time via SDP, but we can also just create w_v copies of each $v \in V$, no two copies of the same vertex and copies of different vertices u, v are adjacent if $uv \in E$. This graph is perfect, as it is obtained by replicating vertices in $\overline{\mathcal{G}}$.

If $\omega(\mathcal{G} - S') = \omega(\mathcal{G})$ then we add a maximum clique C in $\omega(\mathcal{G} - S')$ to the collection $C_1, \ldots, C_\ell$ and repeat. Otherwise, we know $\chi(\mathcal{G} - S') = \chi(\mathcal{G}) - 1$ and we have what we wanted.

The process of finding maximum cliques $C_1, \ldots, C_\ell$ will iterate at most n times before an appropriate S' is found. This is because for any independent set S' having 0/1 indicator vector $\mathbf{v}$ that intersects each clique $C_1, \ldots, C_\ell$ and any clique C in $\mathcal{G} - S'$, we have $\mathbf{v}(C_i) = 1$ yet $\mathbf{v}(C) = 0$, so the dimension of the space $\{\mathbf{x} \in \mathbb{R}^v : \mathbf{x}(C_i) = 1, 1 \leq i \leq \ell\}$ drops by 1 in each iteration. ■