CMPUT 675: Topics in Combinatorics and Optimization	**Fall 2016**

Lecture 29 (Nov 21): Matroid Polytopes

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In this lecture, we discuss polytopes whose extreme points are independent sets in a matroid or in the common intersection of two matroids. We also introduce the uncrossing technique, which is quite useful in other combinatorial problems.

Let M be a matroid over a set X, together with a weight function $w : M \to \mathbf{R}$. We can formulate the maximum weight independent set problem over X as a linear program. Specifically, we consider the space of functions $x : X \to \mathbf{R}$, and take the linear program

$$\begin{aligned} \max \quad & \sum_{e \in X} w(e)x(e) \\ \text{s.t.} \quad & x(A) \leq r(A) \qquad \forall A \subset X \\ & x \geq 0 \end{aligned}$$

in particular, the constraints guarantee that $0 \leq x(e) \leq 1$ for all $e \in X$, because $r(\{e\}) \leq 1$, so integral feasible points in the constraint polyhedron correspond to indicator functions over X. What's more, integral points also correspond to independent sets in the matroid. If $x = \chi_B$ (i.e. is the indicator vector of a subset $B \subseteq X$) lies in the polytope above, then $|B| = x(B) \leq r(B) \leq |B|$, so $r(B) = |B|$, and so B is independant. Thus the corresponding integer linear programming problem for this LP corresponds exactly to the maximal independent set problem.

Summarizing the discussion so far, let $\mathcal{P}_\mathcal{M} = \{x \in \mathbf{R}^X_{\geq 0} : x(A) \leq r(a) \text{ for each } A \subseteq X\}$. We have shown the following:

Lemma 1 *$\mathcal{P}_\mathcal{M} \subseteq [0,1]^X$ and integral points of $\mathcal{P}_\mathcal{M}$ are exactly the indicator vectors of independent sets in $\mathcal{M}$.*

It turns out that (surprise surprise!) all extreme points are integral. Our proof will use an interesting technique known as the uncrossing method, which shows there is a well-structured set of tight constraints that allow us to conclude an extreme point is integral. We note that the constraint matrix of $\mathcal{P}_\mathcal{M}$ is itself not totally unimodular. For example, the constraints corresponding to subsets $A = \{1,2\}, B = \{2,3\}$ and $C = \{1,3\}$ for a matrix that is not totally unimodular.

Theorem 1 *Let $x \in \mathcal{P}_\mathcal{M}$ be an extreme point. Then x is integral.*

Proof. For $A \subseteq X$ we will let $\chi_A \in \mathbf{R}^X$ denote the $\{0,1\}$-indicator vector of A. The proof breaks into two main steps. First, we show there is a well-structured basis of tight constraints for x. Then we show, despite the fact that the constraint matrix of $\mathcal{P}_\mathcal{M}$ is not totally unimodular, this matrix for this particular basis of tight constraints is totally unimodular so x must be integral.

We assume $x_i > 0$ for each $i \in X$. If not, then the restriction of x to $X - \{i\}$ can easily be checked to be an extreme point of $\mathcal{P}_{\mathcal{M}-i}$ and it suffices to prove all extreme points of this polytope are integral.

We say $\mathcal{C} \subseteq 2^X$ is a **chain** if $A, B \in \mathcal{C}$ implies either $A \subseteq B$ or $B \subseteq A$. So the items in a chain form a "nested family". Not that a chain is laminar, but not necessarily the other way around.

Let $\mathcal{C}$ to be the largest chain only containing subsets A for which we have a tight constraint, $x(A) = r(A)$, and such that the corresponding vectors χ_A are independent. We claim that the χ_A span $\mathbf{R}^X$. To that end, we let span($\mathcal{C}$) be shorthand for span($\{\chi_A : A \in \mathcal{C}\}$).

Suppose $B \subseteq X$ is such that $r(B) = x(B)$ yet χ_B is not in span($\mathcal{C}$). Among such B, choose one that minimizes the size of:

$$\tau(B) = \{A \in \mathcal{C} : A \not\subset B \text{ and } B \not\subset A\}$$

Note that we must have $\tau(B)$ must contain at least one element, or else $\mathcal{C} \cup \{B\}$ would be a chain of linearly independent vectors which is larger than $\mathcal{C}$.

Let $T \in \tau(B)$, so $B \notin T$ and $T \notin B$. Note by submodularity of the rank,the fact that both B and T correspond to tight constraints, and because x satisfies all constraints of $\mathcal{P}_\mathcal{M}$ we have

$$\begin{aligned} r(B) + r(T) &= x(B) + x(T) \\ &= x(B \cup T) + x(B \cap T) \\ &\leq r(B \cup T) + r(B \cap T) \\ &\leq r(B) + r(T) \end{aligned}$$

So equality must hold: $x(B\cup T)+x(B\cap T) = r(B\cup T)+r(B\cap T)$. Since $x(B\cup T) \leq r(B\cup T)$, $x(B\cap T) \leq r(B\cap T)$, we have both $x(B \cup T) = r(B \cup T)$, $x(B \cap T) = r(B \cap T)$.

We will show $\tau(B\cup T), \tau(B\cap T) \subsetneq \tau(B)$. By our choice of B, this means $\chi_{B\cup T}, \chi_{B\cap T} \in$ span($\mathcal{C}$). This is clearly in contradiction to our assumption that $\chi_B \notin$ span($\mathcal{C}$) because

$$\chi_B = \chi_{B\cup T} + \chi_{B\cap T} - \chi_T \in \text{span}(\mathcal{C})$$

so all that remains is to prove $\tau(B \cup T), \tau(B \cap T) \subsetneq \tau(B)$.

This is most easily shown by drawing out Venn diagrams, but we'll also provide a textual description.

Consider S in $\mathcal{C} - \{T\}$. Then either $S \subset T$, or $T \subset S$. If $S \subset T$, then $S \subset B \cup T$ and so $S \notin \tau(B \cup T)$. If $T \subset S$, then $B \cap T \subset S$, so $S \notin \tau(B \cap T)$. If $S \in \tau(B \cup T)$ but $S \notin \tau(B)$, then we would have to have $T \subset S$. This also shows $S \subset B$, because if $B \subset S$, then $B \cup T \subset S$. But then $T \subset B$, which is a contradiction. Similarly, if $S \in \tau(B \cap T)$ but $S \notin \tau(B)$, then $S \subset T$ and $B \subset S$, so $B \subset T$, another contradiction. We finish this conclusion by noticing that $T \notin \tau(B \cap T), \tau(B \cup T)$ by obvious relations.

Thus we conclude that $\mathcal{C}$ forms a basis of $\mathbf{R}^X$.

Now our homework comes in handy. Consider the laminar constrainted matching problem, over the Bipartite graph whose left vertices are subsets of X, and with only one right vertex, with an edge between every left vertex. We consider $L_L = \mathcal{C}$ as a laminar family over the left vertices, and $L_R = \emptyset$, with $b_A = r(A)$ for $A \in \mathcal{C}$. In Assignment 3, you showed the constraint matrix for this LP is totally unimodular. But the constraints corresponding to the left-side of the graph are precisely the constraints of $\mathcal{P}_\mathcal{M}$ corresponding to sets $\mathcal{C}$. Thus, x is integral.

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29.0.1 Separating the Constraints

There are exponentially many constraints, so one naturally wonders if it is possible to separate them in polynomial time. Indeed this is the case, but it amounts to determining if the minimum of the following submodular function is negative or not:

$$f : 2^X \to \mathbf{R} \text{ where } f(A) = r(A) - x(A).$$

It is possible to minimize a submodular function in polynomial time. We will sketch the details in a later lecture.

29.1 Base Polytope

One can easily prove $\mathcal{P}_\mathcal{M} \cap \{x \in \mathbf{R}^X : x(X) = r(X)\}$ is integral by showing there is a chain that forms a basis of tight constraints that includes X itself. For example, let $\mathcal{C}$ be the largest such chain and use uncrossing to arrive at a contradiction under the assumption that $\mathcal{C}$ does not form a basis.

29.2 Matroid Intersection Polytope

The matroid intersection problem can be formulated as an LP with integral extreme points in a very similar manner. We consider two matroids M and N, take the same solution space as the maximal independent set LP, and find solutions such that

$$\begin{array}{rll} \max & \sum_{e \in X} w(e)x(e) & \\ \text{s.t.} & x(A) \leq r_M(A) & \forall A \subset X \\ & x(A) \leq r_N(A) & \forall A \subset X \\ & x \geq 0 & \end{array}$$

Theorem 2 *Extreme points of this linear program are integral.*

The proof is mostly identical to the case of one matroid, we just sketch the details. The extreme points of this set are integral, which can be verified from the uncrossing technique of the last problem. Here we will end up with a laminar family of tight constraints $\mathcal{C}_M$ and $\mathcal{C}_N$ over each matroid, rather than a chain, prove that $\text{span}(\mathcal{C}_M \cup \mathcal{C}_N)$ is full, and then use the laminar familiies $L_L = \mathcal{C}_M$, $L_R = \mathcal{C}_N$ to prove integrality as the two chains correspond to submatrices of the same laminar-constrained matching problem from the homework.

Our homework problem does not generalize to intersections of three or more matroids, and for good reason! The matroid intersection problem for more than 2 matroids is NP complete. For example, in a directed graph we can have one matroid assert each vertex has indegree at most 1, another assert each vertex has outdegree 1, and one final matroid assert the underlying set of undirected edges is acyclic. Then there is a set of size $|V| - 1$ that is independent in all matroids if and only if there is a Hamiltonian path in the directed graph.