Lecture 28 (Nov 18): Matroid Intersection and Matroid Partition

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28.1 Matroid Intersection

The Matroid Intersection problem is to find the largest common independent set $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$ when $\mathcal{M}_1 = (X, \mathcal{I}_1)$, $\mathcal{M}_2 = (X, \mathcal{I}_2)$ are matroids. The algorithm to solve the problem is given in the last lecture.

Recall that the heart of the algorithm involves a graph $\mathcal{G}_Y$ where $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$. We defined the following:

- $\mathcal{G}_Y$ is a directed bipartite graph with vertices X. One side is Y and the other is $X - Y$.
- For each $a \in Y, b \in X - Y$ we add directed edge ab if $Y - a + b \in \mathcal{I}_1$.
- For each $a \in Y, b \in X - Y$ we add directed edge ba if $Y - a + b \in \mathcal{I}_2$.
- $Y_1 = \{x \in Y - X : Y \cup \{x\} \in \mathcal{I}_1\}$
- $Y_2 = \{x \in Y - X : Y \cup \{x\} \in \mathcal{I}_2\}$

If there is a path from Y_1 to Y_2 in $\mathcal{G}_Y$, we let P be the vertices on a shortest $Y_1 - Y_2$ path. We will show $Y' := (Y - P) \cup (P - Y) \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $|Y'| = |Y| + 1$, so we have "augmented" Y to a larger set. We will also show that if there is no $Y_1 - Y_2$ path, then Y is already a maximum-size set in $\mathcal{I}_1 \cap \mathcal{I}_2$.

We use the notation above. In particular, Y' is obtained by alternating Y along a shortest $Y_1 - Y_2$ path with vertex set P.

Lemma 1 $|Y'| = |Y| + 1$.

Proof. P is a path that alternates between sides $X - Y$ and Y. It starts and ends in $X - Y$, so $|P \cap (X - Y)| = |P \cap Y| + 1$. Thus, $|Y'| = |Y| + 1$. ∎

Lemma 2 $Y' \in \mathcal{I}_1 \cap \mathcal{I}_2$.

Proof. Let $P = b_0, a_1, b_1, \cdots, a_k, b_k$ be a shortest $Y_1 - Y_2$ path.

We use the following notation. For $Z \subseteq X$ we let $r_i(Z)$ denote the rank of Z in $\mathcal{M}_i$. If $Z \in \mathcal{I}_i$ and $x \in X$ is such that $Z + x \notin \mathcal{I}_i$ we let $C_i(Z, x)$ denote the unique circuit of matroid $\mathcal{M}_i$ contained in $Z + x$ (c.f. the previous lecture).

Note $Y + b_0 \in \mathcal{I}_1$. Further, for each $1 \leq i \leq k$ we have $Y + b_0 + b_i \notin \mathcal{I}_1$, otherwise $b_i \in Y_1$ and the subpath of P starting at b_i would be a shorter $Y_1 - Y_2$ path. Also note $Y + b_i \notin \mathcal{I}_1$, again for the same reason. So, for each $1 \leq i \leq k$ we have

$$C_1(Y, b_i) = C_1(Y + b_0, b_i).$$

Claim 1 *For all* $1 \leqslant i \leqslant k$*,*
i) $a_i \in C_1(Y + b_0, b_i)$*,*
ii) $a_i \notin C_1(Y + b_0, b_i)$ *for* $i < j \leq k$*.*

Proof. For i) we know $a_i \in C_1(Y, b_i)$ because $a_i b_i$ is an edge (i.e. $Y + b_i - a_i \in \mathcal{I}_1$).
So $a_i \in C_1(Y + b_0, b_i)$.

We prove ii) by induction.

For **ii**), otherwise $P^{'} = b_0, a_1, b_1, \cdots, a_j, b_j, \cdots, a_k, b_k$ is a shorter path. ∎

Returning to the proof of Lemma 2, we first prove $Y' \in \mathcal{I}_1$. In particular, we prove by induction on $0 \leq i \leq k$ that $Y^{(i)} \in \mathcal{I}_1$ where

$$Y^{(i)} = Y - \{a_1, \cdots, a_i\} \cup \{b_0, \cdots, b_i\} \in \mathcal{I}_1.$$

For $i = 0$, we already know $Y^{(0)} = Y + b_0 \in \mathcal{I}_1$.

Inductively, for $i > 0$ we note $Y^{(i)} = Y^{(i-1)} - a_i + b_i$ and, by induction, we know $Y^{(i-1)} \in \mathcal{I}_1$. If we have $Y^{(i-1)} + b_i \in \mathcal{I}_1$ then we are done already (removing a_i would leave it in $\mathcal{I}_1$).

Otherwise, let $\overline{C} := C_1(Y^{(i-1)}, b_i)$. We show $a_i \in \overline{C}$. Note, by ii) in the claim, that $C_1(Y + b_0, b_i) \subseteq Y^{(i-1)} + b_i$. So, by uniqueness of circuits, $C_1(Y + b_0, b_i) = \overline{C}$. By i) in the claim, we know $a_i \in C_1(Y + b_0, b_i)$, i.e. $a_i \in \overline{C}$.

A symmetric argument (starting from $i = k$ and going down to 0) shows $Y' \in \mathcal{I}_2$.

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28.2 Min-Max Relationship

Theorem 1 *For any two matroids* $\mathcal{M}_1 = (X, \mathcal{I}_1)$ *and* $\mathcal{M}_2 = (X, \mathcal{I}_2)$ *with rank functions* r_1 *and* r_2 *respectively, we have:*

$$\max_{Y \in \mathcal{I}_1 \cap \mathcal{I}_2} |Y| = \min_{Q \subseteq X} r_1(X - Q) + r_2(Q).$$

Furthermore, for any Y *such that* $\mathcal{G}_Y$ *has no* $Y_1 - Y_2$ *path (using the notation from the algorithm) we have that* Y *is a maximum-cardinality set in* $\mathcal{I}_1 \cap \mathcal{I}_2$*.*

The latter statement in the theorem says when the algorithm above fails to find a $Y_1 - Y_2$ path then Y is already has maximum size.

Proof. As with almost all min/max relationships, we start with the easier part showing the "max" side is at most the "min" side. Let $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $Q \subseteq X$. Simply put,

$$|Y| = |Y - Q| + |Y \cap Q| \leq r_1(X - Q) + r_1(X - Q)$$

because $Y - Q \in \mathcal{I}_1$ and is a subset of X and $Y \cap Q \in \mathcal{I}_2$ and is a subset of Q.

To see equality, let $Y \in \mathcal{I}_1 \cap I_2$ be a set such that there is no $Y_1 - Y_2$ path in $\mathcal{G}_Y$. Let R be all vertices reachable from Y_1 in $\mathcal{G}_Y$ (it could be that $R = \emptyset$ if $Y_1 = \emptyset$, i.e. Y is a base in $\mathcal{I}_1$).

Claim 2 $r_2(R) = |Y \cap R|$.

Proof. Otherwise $r_2(R) > |Y \cap R|$. Then there is some $y \in R - Y$ such that $(Y \cap R) + y \in \mathcal{I}_2$. Note $Y + y \notin \mathcal{I}_2$, otherwise $y \in R \cap Y_2$ which contradicts the fact there is not $Y_1 - Y_2$ path in $\mathcal{G}_Y$. It also cannot be that $C_2(Y, y) \cap (Y - R) = \emptyset$, otherwise the entire circuit would be contained in $(Y \cap R) + y$ which is impossible. But then yx is an edge for some $x \in C_2(Y, y) \cap (Y - R)$, contradicting the fact that y is reachable but x is not. ∎

In a similar way, one can prove $r_1(X - R) = |Y - R|$. Thus,

$$|Y| = |Y - R| + |Y \cap R| = r_1(X - R) + r_2(R).$$

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28.3 Matroid Partition

Let $\mathcal{M}_i(X, \mathcal{I}_i)$ be matroids for $1 \leq i \leq k$. Note they are all over a common ground set. A **partitionable** subset of X is a set Y such that Y can be expressed as the disjoint union of sets $Y_1 \in \mathcal{I}_1, \ldots, Y_k \in \mathcal{I}_k$. Naturally, we are interested in finding the largest partitionable subset of X.

Application

Matroid Colouring: Partition X into the fewest independent sets (i.e. determine the smallest k for which X is itself partitionable when $\mathcal{M}_i(X, \mathcal{I}_i)$ are the same matroid).

Base Packing: Find the largest collection of pairwise-disjoint bases $B_1, \ldots, B_k$. For example, how many edge-disjoint spanning trees can you find? To find this, pick the largest k such that the largest partitionable set between the k identical matroids has size $k \cdot \text{rank}(X)$.

Maker/Breaker: Two players take turns removing edges from an undirected graph. Player 2 wins if the edges they remove contains a spanning tree, player 1 wins if the set of edges they remove would disconnect the original graph. So exactly one player wins.

Who has the winning strategy? It turns out player 2 has the winning strategy if and only if the graph contains two edge-disjoint spanning trees. The last exercise has you working out the details.

Reduction to Matroid Intersection Finding the largest partitionable set reduces to matroid intersection. In particular, we consider the following two matroids. over the common ground set $\overline{X} = \{(x, i) : x \in X, 1 \leq i \leq k\}$.

In the first matroid $\overline{\mathcal{M}}$, we have $\overline{Y}$ being independent if for each $1 \leq i \leq k$ we have $\overline{Y}_i := \{x \in X : (x, i) \in \overline{Y}\}$ being a member of $\mathcal{I}_i$. It is easy to verify this forms a matroid and the rank function is

$$\overline{r}(\overline{Y}) = \sum_{i=1}^{k} r_i(\overline{Y}_i)$$

where r_i is the rank function of $\mathcal{M}_i$.

In the second matroid $\overline{\mathcal{M}}'$ we have $\overline{Y} \subseteq \overline{X}$ being independent if for each $x \in X$ we have $|\{i : (x, i) \in \overline{Y}\}| \leq 1$. The rank function is

$$\overline{r}'(\overline{Y}) = |\{x \in X : (x, i) \in \mathcal{Y} \text{ for some } i\}|.$$

That is, the first matroid $\overline{\mathcal{M}}$ selects an independent set in each matroid $\mathcal{M}_i$ and the second matroid $\overline{\mathcal{M}}'$ ensures each item is allocated to at most one of the $\mathcal{M}_i$.

The following correspondence is clear.

Lemma 3 *If $Y = Y_1 \dot\cup \ldots \dot\cup Y_k$ is a partitionable subset of X (i.e. $Y_i \in \mathcal{I}_i$) then $\{(x,i) : 1 \le i \le k, x \in Y_i\}$ is independent in both $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}'$. Conversely, if $\overline{Y}$ is independent in both $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}'$ then $Y = \cup_i \overline{Y}_i$ is partitionable with the obvious partition $\{\overline{Y}_i \in \mathcal{I}_i\}_{i=1}^k$.*

So, we can find a maximum-size partitionable $Y \subseteq X$ by finding a maximum-size $\overline{Y} \subseteq \overline{X}$ that is independent in both $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}'$ using the matroid intersection algorithm. Furthermore, the running time is polynomial (assuming the independence oracles for each $\mathcal{M}_i$ runs in polynomial time).

Finally, using the min/max relationship for matroid intersection we can get min/max relationship for the largest partitionable set.

Theorem 2

$$\max_{\substack{Y \subseteq X \ s.t. \\ Y \ is \ partitionable}} |Y| = \min_{Q \subseteq X} |X - Q| + \sum_{i=1}^{k} r_i(Q)$$

where r_i is the rank function of $\mathcal{M}_i$.

Proof. If $Y = Y_1 \dot\cup \ldots \dot\cup Y_k$ is partitionable (with $Y_i \in \mathcal{I}_i$) then for any $Q \subseteq X$

$$|Y| = |Y - Q| + |Y \cap Q| \le |X - Q| + |Y \cap Q| = |X - Q| + \sum_i |Y_i \cap Q| \le |X - Q| + \sum_i r_i(Q)$$

as $Y_i \cap Q$ is an independent (in $\mathcal{M}_i$) subset of Q.

Conversely, let $\overline{Y} \subseteq \overline{X}$ be a maximum common independent set between matroid $\overline{\mathcal{M}}$ and $\overline{\mathcal{M}}'$. Let $\overline{Q} \subseteq \overline{X}$ be such that

$$|\overline{Y}| = \overline{r}(\overline{Q}) + \overline{r}'(\overline{X} - \overline{Q}).$$

Such an $\overline{Q}$ exists by Theorem 1. Let $A = \{x \in X : (x,i) \in \overline{Q} \text{ for each } 1 \le i \le k\}$.

Note $\overline{r}'(\overline{Z})$ is just the number of distinct items of X appearing with at least one index in $\overline{Z}$. Thus, $\overline{r}'(X - Q) = |X - A|$. Further, if we let $\overline{Q}_i = \{x : (x,i) \in \overline{Q}\}$ then $\overline{r}(\overline{Q}) = \sum_i r_i(\overline{Q}_i) \ge r_i(A)$, as $A \subseteq \overline{Q}_i$ for each i.

That is,

$$|\overline{Y}| \ge \sum_{i=1}^{k} r_i(A) + |X - A|$$

as required.

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