Lecture 27 (Nov 16): Matriods: Min-Weight Bases, Intersection

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We begin by discussing some properties of matroids and then present the matroid intersection algorithm. The proof of correctness appeared in the next lecture.

27.1 Matroid Rank and Circuits

Let $M = (X, \mathcal{I})$ be a matroid.

Lemma 1 *Let $A \subseteq X$. Let $\mathcal{I}_A = \{B \subseteq A : B \in \mathcal{I}\}$. Then $M_A = (A, \mathcal{I}_A)$ is a matroid.*

Definition 1 *Let $A \subseteq X$. The **rank** of A is given by $r(A) = \max\{|B| : B \subseteq A, B \in \mathcal{I}\}$.*

Lemma 2 *Let $A, B \subseteq X$. Then*

1. $r(A) \leq |A|$
2. $A \subseteq B \implies r(A) \leq r(B)$
3. $r(A) + r(B) \geq r(A \cup B) + r(A \cap B)$

Proof. Properties 1 and 2 follow directly from the definition of rank. We proceed to prove property 3.

Let $J \subseteq A \cap B$, $J \in \mathcal{I}$ be such that $|J| = r(A \cap B)$. Let $K \subseteq A - B$ be such that $|J| + |K| = r(A)$, $J \cup K \in \mathcal{I}$ (one exists because we can extend any independent set to a base of the matroid restricted to A). Similarly, let $L \subseteq B - A$ be such that $|J| + |K| + |L| = r(A \cup B)$, $J \cup K \cup L \in \mathcal{I}$. By construction and properties of matroids, all of J, K, L must exist. We then have

$$\begin{aligned} r(A) + r(B) &= |J \cup K| + r(B) \\ &\geq |J \cup K| + |J \cup L| \\ &= |J \cup K \cup L| + |J| \\ &= r(A \cup B) + r(A \cap B), \end{aligned}$$

where inequality holds because $J \cup B$ is an independent set and the second-last equality holds since all sets are disjoint. ■

Property 3 is also called the *submodularity* property. Together with properties 1 and 2, property 3 is equivalent to the following statement:

$$\forall B \subseteq A, x \notin A, r(B \cup \{x\}) - r(B) \geq r(A \cup \{x\}) - r(A).$$

This is a diminishing returns property, which is a key feature of submodularity. We will see how this property will be useful shortly.

Definition 2 *A subset $C \subseteq X$ is a **circuit** if $C \notin \mathcal{I}$ but $C - \{x\} \in \mathcal{I}$ for all $x \in C$.*

Lemma 3 *Let $y \in \mathcal{I}$, $x \in X$ such that $Y \cup \{x\} \notin \mathcal{I}$. Then there is a unique circuit $C \subseteq Y \cup \{x\}$, denoted $C(Y, x)$.*

Proof. Since $Y \cup \{x\} \notin \mathcal{I}$, there must exist some circuit $C \subseteq Y \cup \{x\}$ (e.g. pick any minimal dependent subset of $Y \cup \{x\}$). We will now prove this circuit is unique.

Suppose for the sake of contradiction that C_1, C_2 are distinct circuits in $Y \cup \{x\}$. Note that since $Y \in \mathcal{I}$, $x \in C_1, C_2$, and $C_1 \nsubseteq C_2$, and so some $y \in C_1 - C_2$ exists. We will show there exists some circuit $C_3 \subseteq (C_1 \cup C_2) - \{x\} \subseteq Y$ - this will imply $Y \notin \mathcal{I}$.

We have

$$\begin{aligned}
|C_1| - 1 + r(C_1 \cup C_2 - \{x, y\}) + |C_2| - 1 &= r(C_1) + r(C_1 \cup C_2 - \{x, y\}) + r(C_2) \\
&\geq r(C_1) + r(C_1 \cup C_2 - \{y\}) + r(C_2 - \{x\}) \\
&\geq r(C_1 - \{y\}) + r(C_1 \cup C_2) + r(C_2 - \{x\}) \\
&= |C_1| - 1 + r(C_1 \cup C_2) + |C_2| + 1,
\end{aligned}$$

where the two inequalities follow by submodularity of the rank function. Thus, $r(C_1 \cup C_2) \leq r(C_1 \cup C_2 - \{x, y\})$. But $C_1 \cup C_2 - \{x, y\} \subseteq C_1 \cup C_2$, so $r(C_1 \cup C_2) = r(C_1 \cup C_2 - \{x, y\}) = r(C_1 \cup C_2 - \{x\})$. Thus, $C_3 = C_1 \cup C_2 - \{x\}$ is not of full rank and so $C_3 \notin \mathcal{I}$. ■

27.2 Matroid Intersection (Intro)

Given matroids $M_1 = (X, \mathcal{I}_1)$ and $M_2 = (X, \mathcal{I}_2)$ over the common ground set X, the *matroid intersection problem* is to find some $Y \subseteq X$, $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$, such that $|Y|$ is maximized (or certify $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$).

We present an incremental algorithm to solve this problem, that takes as input some $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$ and returns either some $Y' \in \mathcal{I}_1 \cap \mathcal{I}_2$ such that $|Y'| = |Y| + 1$, or determines Y is a maximum-size set in $\mathcal{I}_1 \cap \mathcal{I}_2$. We will see the proof of why this algorithm is correct in the next lecture.

Algorithm 1 MATROID-INTERSECTION Algorithm

Input: Matroids $M_1 = (X, \mathcal{I}_1), M_2 = (X, \mathcal{I}_2)$ and $Y \subseteq X$ such that $Y \in \mathcal{I}_1 \cap \mathcal{I}_2$.
Output: $Y' \subseteq X$ such that $Y' \in \mathcal{I}_1 \cap \mathcal{I}_2$ and $|Y'| = |Y| + 1$, or certify Y is maximum.

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E ← ∅        {Will be a set of directed edges in a graph with vertices X}
for a ∈ Y, b ∈ X − Y do
  if Y − {a} ∪ {b} ∈ I_1 then
    E ← E ∪ {ab}
  end if
  if Y − {a} ∪ {b} ∈ I_2 then
    E ← E ∪ {ba}
  end if
end for
Y_1 ← {x ∈ X − Y : Y ∪ {x} ∈ I_1}
Y_2 ← {x ∈ X − Y : Y ∪ {x} ∈ I_2}
Let G_Y = (Y ∪ (X − Y); E) be a directed bipartite graph
if no Y_1 → Y_2 path exists in G_Y then
  return  "Y is maximum"
else
  P ← vertices in a shortest Y_1 → Y_2 path in G_Y
  return  (Y − P) ∪ (P − Y)
end if
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