CMPUT 675: Topics in Combinatorics and Optimization **Fall 2016**

Lecture 25 (Nov. 4): Weighted Perfect Matching in General Graphs

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In this lecture we discus another version of the matching problem: weighted perfect matching in general graphs. Our goal is to solve this problem in polynomial time using linear programming. Before that we first solve the problem of finding the closest fraction (with bounded denominator) to a fixed fraction which was the last step in solving linear programs in polynomial time using separation oracles.

So, first we will provide an algorithm to solve the problem in polynomial time and then we will provide a linear program to find maximum cost perfect matching in general graphs and finally we will prove that the extreme points in the proposed linear program are integral, although it is not totally unimodular.

25.1 LP for Maximum Weighted Perfect Matching in General Graphs

In this lecture we are going to solve another version of the matching problem which is maximum weighted perfect matching in general graphs using linear programs. In this problem we have a graph $\mathcal{G} = (V; E)$ and a cost function $c : E \to R$ (costs could be negative and we denote the cost of edge $e \in E$ by c_e) and our goal is to find a perfect matching M with maximum $c(M)$. In our linear program for each edge $e \in E$ we consider a variable x_e:

$$
\begin{aligned}
\text{maximize :} \quad & \sum_{e \in E} c_e x_e \\
s.t. \quad & x(\delta(v)) = 1 \qquad \forall \quad v \in V \\
& x(E(S)) \leq \frac{|S| - 1}{2} \qquad \forall \quad S \subseteq V, |S| \ odd \\
& x_e \geq 0 \qquad \forall \quad e \in E
\end{aligned}
\tag{25.1}
$$

Here, as before, $E(S)$ is the set of edges which have both endpoints in S. Note that $|V|$ should be even, otherwise we cannot have $x(\delta(v)) = 1$ for all vertices and $x(E(V)) \leq \frac{|V|-1}{2}$ at the same time.

There is another equivalent linear program for the problem:

$$
\begin{aligned}
\text{maximize :} \quad & \sum_{e \in E} c_e x_e \\
s.t. \quad & x(\delta(v)) = 1 \qquad \forall \quad v \in V \\
& x(\delta(S)) \geq 1 \qquad \forall \quad S \subseteq V, |S| \ odd \\
& x_e \geq 0 \qquad \forall \quad e \in E
\end{aligned}
\tag{25.2}
$$

These two are equivalent because for each set of vertices $S \subseteq V$ with odd size we have:

$$x(E(S)) = \frac{\sum_{v \in S} x(\delta(v)) - x(\delta(S))}{2} = \frac{\sum_{v \in S} 1 - x(\delta(S))}{2} = \frac{|S| - x(\delta(S))}{2} \tag{25.3}$$

So we have:

$$x(E(S)) \leq \frac{|S|-1}{2} \qquad \Longleftrightarrow \qquad x(\delta(S)) \geq 1 \tag{25.4}$$

So we can separate the constraints in polynomial time in the following way. Clearly we can check nonnegativity and the degree constraints in polynomial time. To determine if a constraint of the form $x(E(S)) \leq \frac{|S|-1}{2}$ is violated, we compute the minimum capacity odd cut (using capacities x) to see if $x(\delta(S)) \geq 1$ for all $S \subseteq V, |S|$ odd. You did this on the second assignment by showing some edge of a Gomery-Hu tree corresponds to a minimum-capacity odd cut.

In the following we will prove that the extreme points of the first linear program are integral. This is true even though the constraint matrix is not totally unimodular (exercise: why is it not?).

We do this in two major steps.

Step 1)

Lemma 1 *Any extreme point $\overline{x} \in \{x \in R^{|E|}_{\geq 0}, x(\delta(v)) = 1\}$ has half integral components: $x \in \{0, \frac{1}{2}, 1\}^{|E|}$. Furthermore, the collection of edges e with $x_e = \frac{1}{2}$ form a collection of vertex-disjoint cycles of odd length.*

Proof. Assume that $\overline{x}$ is an extreme point of $\{x \in R^{|E|}_{\geq 0}, x(\delta(v)) = 1\}$. With a little work we can assume that for each $e \in E$ we have $\overline{x}_e \in (0,1)$ (otherwise we can remove such an edge and, in the case it had x-value 1, the corresponding vertices and incident edges). This means that each $v \in V$ has at least two edges: $deg(v) \geq 2$, and so we have at least $|V|$ edges: $|E| \geq |V|$. On the other hand, the number of non-zero variables in any basic solution is at most the number of tight constraints that are not non-negativity constraints so we now have $|E| \leq |V|$.

With these two observations we can say that we have exactly $|V|$ edges and each $v \in V$ has $deg(v) = 2$, and so $\mathcal{G}$ is a collection of some cycles. Note that $\mathcal{G}$ could not have any even cycles, otherwise it could be written as a convex combination of two other points (we have seen this before) and so $\overline{x}$ is not an extreme point. So $\mathcal{G}$ is a collection of some odd cycles. Consider an odd cycle $C = \{e_1, e_2, ..., e_{2k+1}\}$. Based on the constraints for each $1 \leq i \leq 2k$ we have: $\overline{x}_{e_i} + \overline{x}_{e_{i+1}} = 1$. This means that we have: $\{\overline{x}_{e_1}, \overline{x}_{e_2}, ..., \overline{x}_{e_{2k+1}}\} = \{\alpha, 1-\alpha, ..., \alpha\}$. Also we should have $\overline{x}_{e_1} + \overline{x}_{e_{2k+1}} = 1$ and so $\alpha = \frac{1}{2}$, and for each $e \in C$ we have $\overline{x}_e = \frac{1}{2}$. This means that $\overline{x}$ is half-integral. ■

Step 2)
We prove the general result (that extreme points of the matching polytope are integral) by induction on V. For the base case consider $|V| = 2$ (the cardinality of V is even). If $E = \emptyset$ then linear program is infeasible and otherwise we have just one solution which is integral.

So assume that $|V| \geq 4$ and $\overline{x}$ is an extreme point for our linear program. First suppose there is no tight constraint of the form $\overline{x}(E(S)) \leq \frac{|S|-1}{2}$ with $3 \leq |S| \leq V - |3|$. Then all tight constraints are from the system $\{x(\delta(v)) = 1, v \in V\} \cup \{x \geq 0\}$ so $\overline{x}$ is half-integral by Lemma 1.

If there is an edge e such that $\overline{x}_e = \frac{1}{2}$, then it must lie on an odd-length cycle C. By considering $S = C$ we have $\overline{x}(E(S)) = \frac{|S|}{2} > \frac{|S|-1}{2}$, which contradicts feasibility of $\overline{x}$. So in this case $\overline{x}$ is integral.

Now suppose that we have some tight constraint such that $\overline{x}(E(S)) = \frac{|S|-1}{2}$ for some $S \subseteq V$, $3 \leq |S| \leq |V - 3|$. Now consider graphs $\mathcal{G}'$ and $\mathcal{G}''$ obtained from $\mathcal{G}$ by contracting S and $V - S$ respectively, into one vertex. Let

x' denote the restriction of x to $\mathcal{G}'$ and x'' the restriction of x to $\mathcal{G}''$ (a quick check verifies both are feasible for the respective LPs).

By induction, the matching polytopes for $\mathcal{G}'$ and $\mathcal{G}''$ are both integral, so x' and x'' can be expressed as convex combinations of integral extreme points, namely matchings of $\mathcal{G}'$ and $\mathcal{G}''$ respectively:

$$x' = \sum_{M'\text{:matchings in G'}} \lambda'_{M'}.x_{M'}, \qquad \Big(\lambda'_{M'} \geq 0, \qquad \sum_{M'\text{:matchings in G'}} \lambda'_{M'} = 1\Big) \tag{25.5}$$

$$x'' = \sum_{M''\text{:matchings in G''}} \lambda''_{M''}.x_{M''}, \qquad \Big(\lambda''_{M''} \geq 0, \qquad \sum_{M''\text{:matchings in G''}} \lambda''_{M''} = 1\Big) \tag{25.6}$$

Also for each $e \in \delta(S)$ we know that e appears in both $\mathcal{G}'$ and $\mathcal{G}''$ and so we have:

$$\sum_{M'\text{:matchings in G' s.t. } e\in M'} \lambda'_{M'} = \sum_{M''\text{:matchings in G'' s.t. } e\in M''} \lambda''_{M''} = \overline{x}_e \tag{25.7}$$

This means that it is possible to combine these decompositions to a decomposition of $\overline{x}$ into matchings and write $\overline{x}$ as a convex combination of matchings in $\mathcal{G}$. From what you showed in the 3rd exercise, this means $\overline{x}$ can in fact be expressed as a convex combination of integer extreme points. Since it is an extreme point itself, it them must be integral.

Zac's Comment #1
I like to think of combining the two decompositions in the following way. The decomposition over matchings in $\mathcal{G}'$ and $\mathcal{G}''$ can be viewed as a probability distribution over matchings. Sample a matching M' from $\mathcal{G}'$ according to its distribution. Then sample a matching M'' from $\mathcal{G}''$ according to its distribution, but conditioned on the particular edge $e \in \delta(S) \cap M'$ being chosen. Overall this gives a probability distribution over perfect matchings in $\mathcal{G}$ and the marginals across the edges agree with $\overline{x}$.

Zac's Comment #2
Somehow this proof seems strange. We being by assuming $\overline{x}$ is an extreme point, we end up in a situation where we express it as a convex combination of integer points (so it doesn't sound like it is necessarily an extreme point) and then wrap it up again by remembering it was an extreme point again. This is an artifact of the order we proved things in this course. There are slightly less "wandering" proofs, but I wanted to use facts you have already seen. If anything, this is good motivation for studying the "the following are equivalent" statements about polytopes being integral (e.g. look in the KV textbook, Theorem 5.13).