CMPUT 675: Topics in Combinatorics and Optimization **Fall 2016**

Lecture 18 (Oct. 19): LP Duality

Lecturer: Zachary Friggstad *Scribe: Zachary Friggstad*

18.1 Cones and Separation

Definition 1 *A* **cone** *is a convex subset $\mathcal{C} \subseteq \mathbb{R}^n$ satisfying the following. For any $\mathbf{y} \in \mathcal{C}$ and any $\lambda \geq 0$ we also have $\lambda \cdot \mathbf{y} \in \mathcal{C}$.*

So $\mathbf{0} \in \mathcal{C}$ for any nonempty cone $\mathcal{C}$.

The separation theorem restricts in a slightly simpler way for cones.

Theorem 1 (Separation Theorem for Cones) *Let $\mathcal{C} \subseteq \mathbb{R}^n$ be a nonempty closed cone. Then for any $\mathbf{x} \notin \mathcal{C}$ there is some $\mathbf{a} \in \mathbb{R}^n$ such that $\mathbf{a}^T \cdot \mathbf{x} < 0$ and $\mathbf{a}^T \cdot \mathbf{y} \geq 0$ for each $\mathbf{y} \in \mathcal{C}$.*

Proof. Let $\mathbf{a}$ and v be as in the separation theorem from the last lecture. Note $\mathbf{0} \in \mathcal{C}$ so $0 = \mathbf{a}^T \cdot \mathbf{0} \geq v$. We also claim no $\mathbf{y} \in \mathcal{C}$ has $\mathbf{a}^T \cdot \mathbf{y} < 0$, otherwise we can scale $\mathbf{y}$ by an arbitrarily large λ to get $\mathbf{a} \cdot (\lambda \cdot \mathbf{y}) < v$ which is impossible because $\lambda \cdot \mathbf{y} \in \mathcal{C}$ as well.

So we then have $\mathbf{a}^T \cdot \mathbf{y} \geq 0$ for all $y \in \mathcal{P}$ and also $\mathbf{a}^T \cdot \mathbf{x} < v \leq 0$. ■

18.2 Farkas Lemma

This lemma describes a dichotomy for linear inequalities. Either the system has a feasible solution or there is a "dual" solution that quickly demonstrates infeasibility of the original. Ultimately, this will be seen to be a relatively simple consequence of the separation theorem.

Theorem 2 (Farkas Lemma) *Let $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$. Then exactly one of the following holds*

1. *There is some $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ such that $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$.*
2. *There is some $\mathbf{y} \in \mathbb{R}^m_{\geq 0}$ such that $\mathbf{A}^T \cdot \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^T \cdot \mathbf{y} = -1$.*

Proof. We first show both cannot happen. Suppose such $\mathbf{x}$ and $\mathbf{y}$ exist as in both parts of the statement. On one hand

$$\mathbf{y}^T \cdot (\mathbf{A} \cdot \mathbf{x}) \leq \mathbf{y}^T \cdot \mathbf{b} = -1.$$

On the other hand

$$(\mathbf{y}^T \cdot \mathbf{A}) \cdot \mathbf{x} \geq \mathbf{0}^T \cdot \mathbf{x} = 0.$$

But this shows $0 \leq \mathbf{y}^T \cdot \mathbf{A} \cdot \mathbf{b} \leq -1$, which is impossible.

We now show at least one of these two cases holds. Let $\mathbf{M} = (\mathbf{A} \mid \mathbf{I}_m)$ (i.e. extend $\mathbf{A}$ by adding the adjacency matrix to the right). Let

$$\text{cone}(\mathbf{M}) = \left\{ \mathbf{M} \cdot \begin{pmatrix} \mathbf{x} \\ \mathbf{s} \end{pmatrix} : \begin{pmatrix} \mathbf{x} \\ \mathbf{s} \end{pmatrix} \in \mathbb{R}^{n+m}_{\geq 0} \right\}.$$

It is easy to see $\text{cone}(\mathbf{M})$ is a closed and nonempty cone.

1. If $\mathbf{b} \in \text{cone}(\mathbf{M})$ then we know there is some $\mathbf{x} \in \mathbb{R}^n_{\geq 0}, \mathbf{s} \in \mathbb{R}^m_{\geq 0}$ such that $\mathbf{A} \cdot \mathbf{x} + \mathbf{I}_m \cdot \mathbf{s} = \mathbf{b}$, so $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$.
2. If $\mathbf{b} \notin \text{cone}(\mathbf{M})$ then, by Theorem 1, there is some $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{y}^T \cdot \mathbf{b} < 0$ and $\mathbf{y}^T \cdot \mathbf{M} \geq \mathbf{0}$. Thus, $\mathbf{I}_m \cdot \mathbf{y} \geq \mathbf{0}$ so $\mathbf{y} \in \mathbb{R}^m_{\geq 0}$. Also because $\mathbf{y}^* \cdot \mathbf{M} \geq \mathbf{0}$ we have $\mathbf{A}^T \cdot \mathbf{y} \geq \mathbf{0}$. Finally, by scaling $\mathbf{y}$ by a strictly positive amount if necessary (which preserves $\mathbf{y} \geq \mathbf{0}$ and $\mathbf{A}^T \cdot \mathbf{y} \geq)$) we have $\mathbf{b}^T \cdot \mathbf{y} = -1$.

■

18.3 Strong Duality of Linear Programming

We use the Farkas' Lemma in a fairly straightforward way to conclude that strong duality holds in linear programming. We note that we could have devised a more direct proof from the separation theorem, but Farkas' lemma is useful in more contexts than this so it is helpful to prove it as a stepping stone to duality.

We set up notation first. Let $\mathbf{A} \in \mathbb{R}^{m \times n}, \mathbf{b} \in \mathbb{R}^m$ and $\mathbf{c} \in \mathbb{R}^n$. Consider the polyhedra

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\} \text{ and } \mathcal{D} = \{\mathbf{y} \in \mathbb{R}^m_{\geq 0} : \mathbf{A}^T \cdot \mathbf{y} \geq \mathbf{c}\}.$$

The **primal LP** is

$$\max_{\mathbf{x} \in \mathcal{P}} \mathbf{c}^T \cdot \mathbf{x}$$

and the **dual LP** is

$$\min_{\mathbf{y} \in \mathcal{D}} \mathbf{b}^T \cdot \mathbf{y}.$$

Theorem 3 *The primal LP has an optimal solution if and only if the dual LP has an optimal solution. In the case they have optimal solutions, their values are the same:*

$$\max_{\mathbf{x} \in \mathcal{P}} \mathbf{c}^T \cdot \mathbf{x} = \min_{\mathbf{y} \in \mathcal{D}} \mathbf{b}^T \cdot \mathbf{y}.$$

Proof. We first suppose the dual has an optimal solution $\mathbf{y}^*$ with value $v^* = \mathbf{b}^T \cdot \mathbf{y}^*$.

Let $\mathbf{M} = \begin{pmatrix} -\mathbf{c}^T \\ \mathbf{A} \end{pmatrix} \in \mathbb{R}^{(m+1) \times n}$ and $\mathbf{d} = \begin{pmatrix} -v^* \\ \mathbf{b} \end{pmatrix} \in \mathbb{R}^{m+1}$. Suppose there is some $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ such that $\mathbf{M} \cdot \mathbf{x} \leq \mathbf{d}$. Restricting this to the first row, we have $-\mathbf{c}^T \cdot \mathbf{x} \leq -v^*$ or $\mathbf{c}^T \cdot \mathbf{x} \geq v^*$. The remaining rows show $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$. So in fact $\mathbf{x} \in \mathcal{P}$ and, by weak duality, it must also be $\mathbf{c}^T \cdot \mathbf{x} \leq v^*$. That is, $\mathbf{c}^T \cdot \mathbf{x} = v^*$ as well.

Otherwise, by Farkas Lemma, there is some $\begin{pmatrix} y_0 \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{m+1}_{\geq 0}$ such that $\mathbf{M}^T \cdot \begin{pmatrix} y_0 \\ \mathbf{y} \end{pmatrix} \geq \mathbf{0}$ and $\mathbf{d}^T \cdot \begin{pmatrix} y_0 \\ \mathbf{y} \end{pmatrix} = -1$ (here we are viewing $y_0 \in \mathbb{R}_{\geq 0}$ and $\mathbf{y} \in \mathbb{R}^m_{\geq 0}$).

Two cases:

- If $y_0 = 0$, then this really says $\mathbf{A}^T \cdot \mathbf{y} \geq \mathbf{0}$ and $\mathbf{b}^T \cdot \mathbf{y} = -1$, so $\mathbf{A}^T(\mathbf{y}^* + \mathbf{y}) \geq \mathbf{c}$ and $\mathbf{b}^T \cdot (\mathbf{y}^* + \mathbf{y}) = v^* - 1$, which contradicts the fact that y^* is an optimal dual solution with value v^*.

- If $y_0 > 0$, then let $\mathbf{y}' = \frac{1}{y_0} \cdot \mathbf{y} \geq \mathbf{0}$.

  Now, $\mathbf{M}^T \cdot \begin{pmatrix} y_0 \\ \mathbf{y} \end{pmatrix} \geq \mathbf{0}$ means $\mathbf{A}^T \cdot \mathbf{y}' \geq \mathbf{c}$. That is, $\mathbf{y}'$ is a feasible LP solution.

  Then $\mathbf{d}^T \cdot \begin{pmatrix} y_0 \\ \mathbf{y} \end{pmatrix} = -1$ means $\mathbf{b}^T \cdot \mathbf{y}' = v^* - \frac{1}{y_0} < v^*$, contradicting optimality of y^*.

So our assumption that the second case of Farkas Lemma applies to $\mathbf{M}, \mathbf{d}$ was incorrect. That is, the primal LP has an optimal solution with value v^* as well.

To conclude we should show that if the primal LP has an optimal value, then so to does the dual LP and these values are the same. The proof is essentially the same, let $\mathbf{x}^*, v^*$ be an optimal solution and its value to the primal LP. Consider the matrix $\mathbf{M} = \begin{pmatrix} \mathbf{b}^T \\ -\mathbf{A}^T \end{pmatrix}$ and the vector $\mathbf{d} = \begin{pmatrix} v^* \\ -\mathbf{c} \end{pmatrix}$ and apply Farkas Lemma in a nearly identical way.

■

18.4 Final Details

We conclude our discussion of the general theory of LP duality by noting a few details. Strong duality shows if the primal or dual have an optimal solution then the other must as well. But what other combinations are possible?

We summarize all possibilities below:

- If the primal or dual has an optimal solution, then the other does as well (and they have equal value).
- If the primal is unbounded, then the dual must be infeasible because weak duality shows any dual solution places a bound on the value of any primal solution. Similarly, if the dual is unbounded then the primal is infeasible.
- It turns out it is possible for both to be infeasible:

$$\begin{array}{rrcr} \text{maximize}: & x_1 + x_2 & & \\ \text{subject to}: & -x_1 & \leq & 1 \\ & x_2 & \leq & -1 \\ & x_1, x_2 & \geq & 0 \end{array} \qquad \begin{array}{rrcr} \text{minimize}: & -y_1 + y_2 & & \\ \text{subject to}: & -y_1 & \geq & 1 \\ & y_2 & \geq & 1 \\ & y_1, y_2 & \geq & 0 \end{array}$$

  Re-evaluate your life choices if you ever end up in this situation :)

Finally, there is the extremely important notion of **complementary slackness**. In the proof below, we consider LPs in standard form but it is trivial to modify the statement for LPs in general form.

Theorem 4 *Let $\overline{\mathbf{x}} \in \mathcal{P}$ and $\overline{\mathbf{y}} \in \mathcal{D}$. Then $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ are both optimal for their respective LPs if and only if both conditions below hold:*

1. *For each $1 \leq j \leq n$ we have $\overline{\mathbf{x}}_j \cdot (\mathbf{A}_j^T \overline{\mathbf{y}} - \mathbf{c}_j) = 0$.*
2. *For each $1 \leq i \leq m$ we have $\overline{\mathbf{y}}_i \cdot (\mathbf{A}_i \overline{\mathbf{x}} - \mathbf{b}_i) = 0$.*

Proof. Recall the weak duality argument:

$$\sum_j \mathbf{c}_j \cdot \overline{\mathbf{x}}_j \leq \sum_j \left(\sum_i \mathbf{A}^T_{j,i} \overline{\mathbf{y}}_i \right) \cdot \overline{\mathbf{x}}_j = \sum_i \left(\sum_j \mathbf{A}_{i,j} \overline{\mathbf{x}}_j \right) \cdot \overline{\mathbf{y}}_i \leq \sum_i \mathbf{b}_i \cdot \overline{\mathbf{y}}_i.$$

The first inequality holds term by term for each j, so it holds with equality if and only if condition 1 from the statement of the theorem holds. Similarly, the second inequality holds with equality if and only if condition 2 holds.

So if both $\overline{\mathbf{x}}, \overline{\mathbf{y}}$ are optimal then strong duality shows equality holds throughout. Conversely, if equality holds throughout then in fact we must have that $\overline{\mathbf{x}}, \overline{\mathbf{y}}$ are both optimal solutions. ■

We note that if only condition 1 or condition 2 holds, then we cannot conclude that either $\overline{\mathbf{x}}$ or $\overline{\mathbf{y}}$ is optimal. It is an all-or-nothing statement.