CMPUT 675: Topics in Combinatorics and Optimization **Fall 2016**

Lecture 15 (Oct. 12): Linear Programming

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In this lecture we continue the discussion about the linear programming. First we will provide a useful lemma, then we will examine the bit complexity of extreme point solutions. Finally we will explore totally unimodular (TUM) matrices and show that if constraint matrix A is TUM and $b \in \mathbb{Z}^m$, our LP has integer extreme points.

15.1 Calculation of extreme point values

First of all we have a definition.

Definition 1 *Let the ith column of $M \in R^{m\times n}$ be denoted by m_i. Then for each column vector $b \in R^m$, we denote the matrix obtained by replacing the ith column of M with b by $M(i;b)$.*

For example, if

$$M = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad b = \begin{pmatrix} 10 \\ 11 \\ 12 \end{pmatrix}$$

then

$$M(2;b) = \begin{pmatrix} 1 & 10 & 3 \\ 4 & 11 & 6 \\ 7 & 12 & 9 \end{pmatrix}.$$

Recall our standard notation when deadline with LPs. Let $\overline{x} \in \mathcal{P} = \{x \in R_{\geq 0}, Ax \leq b\}$ be a feasible solution to our linear program, and also $A^{\overline{x}}$ and $b^{\overline{x}}$ be the matrix and rhs-vector of tight constraints for $\overline{x}$ respectively (so $A^{\overline{x}}\overline{x} = b^{\overline{x}}$). From the previous lecture we know that $\overline{x}$ is an extreme point if and only if $\text{rank}(A^{\overline{x}}) = n$.

We consider some extreme point $\overline{x}$. Let A' be any $n \times n$ submatrix of $A^{\overline{x}}$ with $\text{rank}(A') = n$ (there will be at least one, from basic linear algebra results). Let b' be the corresponding entries of $b^{\overline{x}}$. Recall that $\text{rank}(A') = n \iff \det A' \neq 0$. So

Lemma 1 *For each $1 \leq i \leq n$,*

$$\overline{x}_i = \frac{\det A'(i;b')}{\det A'}.$$

This is exactly Cramer's rule from linear algebra. We call such a full-rank matrix A' and vector b' a **basis representation for the tight constraints**.

15.2 Bit complexity of extreme points

In this section we will show that the bit complexity of representing any extreme point of LP (if it is not infeasible or unbounded) is polynomial to the size of input.

First of all we know that the number of bits required to write an integer k (without considering the low level details) is $\lceil \log k + 1 \rceil \in O(\log k)$. Also the number of bits required to write a fraction $\frac{a}{b} \in \mathbb{Q}$ is $O(\lg a + \lg b)$.

Assuming, without loss of generality, all values are integers
Another important fact is that we can assume that A and b only have integer entries. This is by clearing the denominators in each row. Let Δ denote the maximum bit complexity of the denominators appearing among all entries of A, b, c (so each denominator is at most 2^{Δ}).

Let ℓ_i be the product of all denominators of row A_i and entry b_i. Then constraint $M_i x \leq b_i$ is equivalent to $(\ell_i \cdot M_i)x \leq \ell_i \cdot b_i$. As all entries in this constraint are scaled by at most 2^{Δ}, the total bit complexity in this row increases by $(n+1)\cdot\Delta$ which is polynomial in the input size. We can clear the objective function denominators in the same way, which preserves optimality of solutions and only scales their value by the scalar used to clear denominators. Thus, we assume all given coefficients are integers.

We now proceed to bound the bit complexity of extreme point solutions.

Definition 2 *Let $\mathcal{S}_n$ denote the set of all permutations $\sigma : [n] \to [n]$ of $[n] = \{1, 2, \ldots, n\}$. For each $\sigma \in \mathcal{S}_n$, we let*

$$\mathrm{sgn}(\sigma) = (-1)^{|\{i<j:\sigma(i)>\sigma(j)\}|}.$$

Another view is that $\mathrm{sgn}(\sigma)$ is 1 if the number of odd-length cycles in the cycle decomposition of σ is even, and is -1 if the number of odd-length cycles in the cycle decomposition of σ is odd.

The Leibniz formula for the determinant of an $n \times n$ matrix A is:

$$\det A = \sum_{\sigma} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} A_{i,\sigma(i)}. \tag{15.1}$$

This expresses the determinant as an integer-linear combination of some products of its entries. So we immediately

Lemma 2 *If $A \in \mathbb{Z}^{n\times n}$ then $\det A \in \mathbb{Z}$.*

On the other hand, recall Hadamard's bound for determinants:

$$\forall A \in R^{n\times n}, \quad |\det A| \leq \prod_{i=1}^{n} ||A_i||_2 = \prod_{i=1}^{n} \left(\sqrt{\sum_{j=1}^{n} (A_{ij})^2} \right) \tag{15.2}$$

Now let Δ be the smallest number such that every entry of A and b lies in the range$[-2^{\Delta}, 2^{\Delta}]$. Then by Cramer's rule and Hadamard's bound, we have

Lemma 3 *Let A', b' be a basis representation for the tight constraints of an extreme point $\overline{x}$. Then both the numerator and denominator of x_i are bounded, in absolute value, by abc.*

Proof. On one hand, $\det A'$ is an integer. On the other hand,

$$\det A' \leq \prod_i \sqrt{\sum_j A_{i,j}^2} \leq \prod_i \sqrt{n 2^{2\Delta}} = n^{n/2} 2^{n\Delta}.$$

Thus, the number of bits required to write the denominator of any x_i is $\log \det A' = O(n\Delta \log n)$. A similar argument applied to $\det A'(i;b)$ will also bound the numerator's bit complexity by $O(n\Delta \log n)$. ∎

15.3 Totally Unimodular Matrix

Definition 3 *A matrix $A \in \mathbb{R}^{m\times n}$ is* **totally unimodular (TUM)** *if for every square submatrix $A' \in \mathbb{R}^{k\times k}$:*

$$\det A' \in \{-1, 0, +1\} \tag{15.3}$$

By considering the 1×1 submatrices we see a TUM matrix can only have entries $-1, 0$ or $+1$.

Theorem 1 *If $A \in \mathbb{R}^{m\times n}$ is TUM and $b \in \mathbb{Z}^m$, then every extreme point $\overline{x}$ of $\mathcal{P} = \{x \in \mathbb{R}^n_{\geq 0}; Ax \leq b\}$ is in $\mathbb{Z}^n$.*

Proof of Theorem 1. First we will show that if $A \in R^{m\times n}$ is TUM, then $\left(\frac{A}{I_n}\right) \in R^{(m+n)\times n}$ (which is obtained by adding rows of the identity matrix to A) is TUM too.

Let $A' \in Z^{k\times k}$ be a square submatrix of $\left(\frac{A}{I_n}\right)$ we will prove that $\det A' \in \{-1, 0, +1\}$ by induction on the number of rows from I_n in A' which we denote by $\alpha(A')$.

For the base step we should set $\alpha(A') = 0$. In this case A' is a square submatrix of A which is TUM and so: $\det A' \in \{-1, 0, +1\}$.

Inductively, suppose row i of I_n is a row of A'. If the row in A' is all zeros (A' does not contain ith column of $\left(\frac{A}{I_n}\right)$), then the determinant would be zero. Otherwise say row i' in A' is from row i of I_nDenote the matrix obtained by removing row i' and column i from A' by $A'^{[i',j]}$. Then

$$\det A' = \sum_j \left((-1)^{(i'+j)} . \big(\det A'^{[i',j]}\big) \times A_{i'j} \right) = (-1)^{(i'+i)} . \big(\det A'^{[i',i]}\big) \tag{15.4}$$

Since $A'^{[i',j]}$ is an square submatrix of $\left(\frac{A}{I_n}\right)$ which has $\alpha(A') - 1$ rows from I_n, by induction its determinant is in $\{-1, 0, +1\}$:

$$\det A'^{[i',j]} \in \{-1, 0, +1\} \quad \Rightarrow \quad \det A' \in \{-1, 0, +1\} \tag{15.5}$$

Now suppose A', b' basis representation for the tight constraints of extreme point $\overline{x} \in \mathcal{P}$. Because A' is a square submatrix of TUM matrix $\left(\frac{A}{I_n}\right)$ and has rank n, we have $\det A' = \pm 1$. Also, $A'(i;b) \in \mathbb{Z}^{n\times n}$ because A' is TUM (so all entries are integers) and $b \in \mathbb{Z}^m$. So by Cramer's rule, we see

$$\overline{x}_i = \pm \det A'(i;b) \in \mathbb{Z}.$$

That is, $\overline{x} \in \mathbb{Z}^n$.

∎

This also (essentially) shows the decision problem of determining if an LP has a solution with value at least some given value v lies in NP[1].

[1]For bounded LPs we just use an optimum extreme point as a "yes" certificate in the case the LP value is at least v. For unbounded LPs, we need to work just a tiny bit harder: add $c^T x \geq v$ to the constraints and produce any extreme point.

In the next lecture we will see some problems whose natural LP relaxations have a TUM matrix like the bipartite matching problem.