CMPUT 675: Topics in Combinatorics and Optimization **Fall 2016**

Lecture 10 (Oct. 5): MMCC Algorithm-Matching in General Graph

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10.1 Minimum-Cost Bipartite Matching

In this lecture we want to show the correctness of Minimum Mean Cycle Canceling algorithm which was mentioned in the previous lecture.

For a flow f, let $\text{mean}(f_i)$ denote the minimum ratio $\frac{c(\mathcal{C}_i)}{|\mathcal{C}_i|}$ of a cycle in the residual graph $\mathcal{G}_{f_i}$.

Lef f_i be the flow after iteration i and $\mathcal{C}_i$ be minimum ratio cycle in G_{f_i}. So, $\text{mean}(f_i) := \dfrac{c(\mathcal{C}_i)}{|\mathcal{C}_i|}$. The following was shown in the previous lecture.

$$\text{mean}(f_i) \leq \text{mean}(f_{i+1}) \quad (1)$$

$$\text{mean}(f_i) \leq 2 \cdot \text{mean}(f_{i+m\cdot n}) \quad (2)$$

Let $k := m \cdot n(\lceil \log_2^n \rceil + 1)$

Lemma 1 *For every iteration i where $i + k$ is not the final minimum-cost flow, there exists $e \in \mathcal{C}_i$ such that $e \notin E_{f_j}$, $\forall j \geq i + k$.*

If this holds, then the algorithm will terminate in at most $m \cdot k$ iterations.

Proof.

Observation 1
According to Equation (2) and the definition of k, we have the following:

$$\text{mean}(f_i) \leq 2^{\lceil \log_2^n \rceil + 1} \text{mean}(f_{i+k}) \leq 2n \cdot \text{mean}(f_{i+k})$$

where the last inequality is by using an obvious fact that $\lceil \log_2^n \rceil \geq \log_2^n$.

Observation 2
Let $c'(e) = c(e) - \text{mean}(f_{i+k})$ for all $e \in E_{f_{i+k}}$.

Note: For every $\mathcal{C}$ in $G_{f_{i+k}}$, $c'(\mathcal{C}) = c(\mathcal{C}) - \text{mean}(f_{i+k}) \cdot |\mathcal{C}| \geq 0$; because $\text{mean}(f_{i+k}) \leq \dfrac{c(\mathcal{C})}{|\mathcal{C}|}$.

So, there exists a potential ϕ for $(G_{f_{i+k}}, c')$: we have $c'_\phi(e) \geq 0, \forall e \in E_{f_{i+k}}$.

Observation 3
$0 \leq c'_\phi(e) = c_\phi(e) - \text{mean}(f_{i+k})$ for all $e \in E_{f_{i+k}}$.

By using the definition of $c'_\phi(e)$.

Observation 4
$c_\phi(\mathcal{C}_i) = c(\mathcal{C}_i) = \text{mean}(f_i) \cdot |\mathcal{C}_i| \leq 2n \cdot \text{mean}(f_{i+k}) \cdot |\mathcal{C}_i|$.

This follows from Observation 1. Hence, there is an edge $e^* \in \mathcal{C}_i$ such that $c_\phi(e^*) \leq 2n \cdot \text{mean}(f_{i+k})$.

Observation 5
For this edge $e^* \in E_{f_i}$, we claim $e^* \notin E_{f_{i+k}}$. Assume otherwise, then by Observation 3, $\text{mean}(f_{i+k}) \leq c_\phi(e^*) \leq 2n \cdot \text{mean}(f_{i+k})$ where the last inequality comes from Observation 4. This is a contradiction since $\text{mean}(f_{i+k})$ is a strictly negative (because $i + k$ is not the last iteration).

Claim 1 *For every flow $\overline{f}$ of value γ, $e^* \notin E_{\overline{f}}$ if* $\text{mean}(\overline{f}) \geq \text{mean}(f_{i+k})$.

Proof. Let $\overline{f}$ be a flow of value γ with $e^* \in E_{\overline{f}}$. We will demonstrate a cycle in $G_{\overline{f}}$ with mean cost less than $\text{mean}(f_{i+k})$.

Let $g = \overline{f} - f_{i+k}$ and $\overline{g}$ be the corresponding circulation in $G_{f_{i+k}}$: namely set $\overline{g}(e)$ if $g(e) \geq 0$ and $\overline{g}(e) = g(\overleftarrow{e})$ if $g(e) < 0$. We saw in an earlier lecture that constructing $\overline{g}$ in this manner only puts nonzero flow on edges in $G_{f_{i+k}}$.

Since, $e^* \in E_{\overline{f}} - E_{f_{i+k}}$, then $\overline{g}(\overleftarrow{e^*}) > 0$. This, plus the fact that $\overline{g}$ is a circulation in $G_{f_{i+k}}$ means there is a cycle $\mathcal{C}^*$ in $G_{f_{i+k}}$ with $e^* \in \mathcal{C}^*$. Note

$$c(\mathcal{C}^*) = c_\phi(\mathcal{C}^*) = \sum_{e \in \mathcal{C}^*} c_\phi(e) = c_\phi(\overleftarrow{e^*}) + c_\phi(\mathcal{C}^* - \{\overleftarrow{e^*}\})$$

If we apply Observation 3 on each all edges in $\mathcal{C}^* - \{\overleftarrow{e^*}\}$, we get that $c_\phi(\mathcal{C}^* - \{\overleftarrow{e^*}\}) \geq (|\mathcal{C}^*| - 1) \cdot \text{mean}(f_{i+k})$. Then using Observation 4, we conclude that:

$$c(\mathcal{C}^*) \geq -2n \cdot \text{mean}(f_{i+k}) + (|\mathcal{C}^*| - 1) \cdot \text{mean}(f_{i+k}) > -n \cdot \text{mean}(f_{i+k})$$

By considering $f_{i+k} - \overline{f}$, we see $\overleftarrow{\mathcal{C}^*} = \{\overleftarrow{e} : e \in \mathcal{C}^*\}$ is a cycle in $G_{\overline{f}}$. The above bound means $c(\overleftarrow{\mathcal{C}^*}) < n \cdot \text{mean}(f_{i+k})$. So,

$$\text{mean}(\overline{f}) \leq \frac{c(\overleftarrow{\mathcal{C}^*})}{|\overleftarrow{\mathcal{C}^*}|} \leq \frac{c(\overleftarrow{\mathcal{C}}^*)}{n} < \text{mean}(f_{i+k}).$$

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This also completes the proof of the theorem.

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Using the algorithm to find minimum-ratio cycles in $O(mn^2)$ time plus the bound on the number of iterations, we see that we can find a minimum-cost cycle in time $O(m^3 n^3 \log n)$. By using a faster $O(mn)$ algorithm to find minimum-ratio cycles (in the Korte-Vygen texbook), we can further reduce this running time.

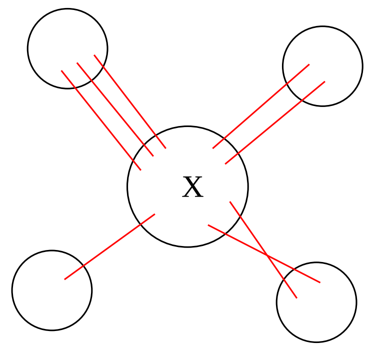

Figure 10.1: A set X and the connected components obtained by deleting X (and all incident edges) from $\mathcal{G}$.

10.2 Matchings in General Graph

We shift focus back to computing maximum matchings. Earlier in the course, we saw how to compute a maximum matching in a bipartite graph in polynomial time. Now we will find maximum matchings in any undirected graph.

Let $\mathcal{G} = (V, E)$ be an undirected graph. For $X \subseteq V$, let $q_{\mathcal{G}}(X)$ be the number of connected components in $\mathcal{G} - X$ with an odd number of vertices, see Figure 10.1.

Definition 1 *Graph $\mathcal{H} = (V', E')$ is factor critical if for all $v \in V'$, $\mathcal{H} - v$ has a perfect matching.*

According to this definition, a graph with just one node is also factor critical.

Theorem 1 *Graph $\mathcal{G}$ has a perfect matching iff $q_{\mathcal{G}}(X) \le |X|$ for every $X \subseteq V$.*

Proof.

We prove this by induction on $|V|$. The case $|V| = 1$ is trivial: there is no perfect matching and by considering $X = \emptyset$ we see $1 = q_{\mathcal{G}}(X) > |X|$.

Suppose $\mathcal{G}$ has perfect matching and let C be an odd component of $\mathcal{G} - X$. Since every vertex in C is matched and $|C|$ is odd, then some vertex of C must be matched with a vertex of X. So it must be $q_{\mathcal{G}}(X) \le |X|$.

Conversely, suppose that for every $X \subseteq V$, $q_{\mathcal{G}}(X) \le |X|$. Then $|V|$ will be even, otherwise if $|V|$ is odd then some connected component of V is an odd component, so $q_{\mathcal{G}}(\emptyset) \ge 1$, which contradicts the assumption $q_{\mathcal{G}}(\emptyset) \le 0$.

Let $even(\mathcal{H})$ and $odd(\mathcal{H})$ be the vertex sets of the even components and odd components of a graph $\mathcal{H} = (V', E')$, respectively. Note, for every $X \subseteq V'$ we have

$$|V'| = |X| + \sum_{C \in even(\mathcal{H}-X)} |C| + \sum_{C \in odd(\mathcal{H}-X)} |C| \equiv |X| + |q_{\mathcal{H}}(X)| \pmod 2.$$

In particular, because $|V|$ is even we have for any $X \subseteq V$ that

$$|X| \equiv q_{\mathcal{G}}(X) \pmod 2. \tag{10.1}$$

Next, let $X \subseteq V$ be any maximal set such that $|X| = q_{\mathcal{C}}(X)$. Such a set exists by looking at the singleton sets: for each $v \in V$, $q_{\mathcal{G}}(\{v\}) \le |\{v\}| = 1$ and according to (10.1), $q_{\mathcal{G}}(\{v\}) \equiv |\{v\}| \equiv 1 \pmod 2$. So, $q_{\mathcal{G}}(\{v\}) = 1$.

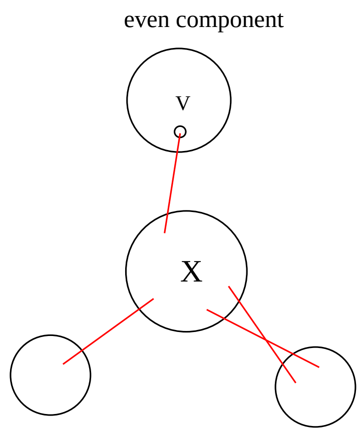

Figure 10.2: Example for the proof of Theorem 1

Claim 2 *$\mathcal{G} - X$ has no even component.*

Proof. We prove this claim by contradiction. Assume that C is an even component of $\mathcal{G} - X$. Pick any v in C. Every odd component of $\mathcal{G} - (X \cup \{v\})$ is still an odd component of $\mathcal{G} - X'$. There will also be at least one more odd component that is a subgraph of $C - \{v\}$ (as $|C - \{v\}|$ is odd). So $q_{\mathcal{G}}(X \cup \{v\}) \geq |X| + 1$. See Figure 10.2.

However, according to our inductive step, $q_{\mathcal{G}}(X \cup \{v\}) = |X| + 1$ which contradicts the maximality of set X. ■

Comment: The remainder of this proof was completed in the subsequent lecture. So don't let the following **q.e.d.** symbol fool you :)

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