CMPUT 675 - Assignment #5
Due Dec 9 by 1pm

Fall 2016, University of Alberta

Pages: 3

Exercise 1)

Marks: 5

Let $\mathcal{G} = (V; E)$ be a connected graph. Consider the following 2-player game. Players alternate turns where each player $i \in \{1, 2\}$ maintains some $F_i \subseteq E$ where $F_1 \cap F_2 = \emptyset$. Initially both F_1 and F_2 are empty. Player 1 plays first.

A play by player i is simply adding some edge of $E - (F_1 \cup F_2)$ to F_i. Once $F_1 \cup F_2 = E$, the game ends. Player 1 wins if $(V; F_2)$ is disconnected, player 2 wins if $(V; F_2)$ is connected.

Here you will show player 2 has a winning strategy if and only if there are two edge-disjoint spanning trees. To get you started, note that the specialization of the matroid partition min/max theorem proven in class shows $\mathcal{G}$ has two disjoint spanning trees if and only if

$$2 \cdot (|V| - 1) \leq \min_{F \subseteq E} |E - F| + 2 \cdot r(F).$$

- Show that if $\mathcal{G}$ has two edge-disjoint spanning trees then no matter what player 1 chooses in their turn player 2 can maintain that either F_2 already contains a spanning tree or there are two spanning trees T_1, T_2 such that $T_1 \cap T_2 = F_2$ and both $T_1 - F_2, T_2 - F_2 \subseteq E - F_1$ (i.e. are not yet grabbed by any player).
- For a partition $\pi \subseteq 2^V$ of V into $|\pi|$ parts let $\partial(\pi)$ be all edges in E that have endpoints in different parts. Show $\mathcal{G}$ has two edge-disjoint spanning trees if and only if $|\partial(\pi)| \geq 2(|\pi| - 1)$ for all partitions π of V.
  Hint: For the "harder" direction, start by showing the minimum of the expression in the min/max relationship recalled above is achieved at some set F where $E - F$ is of the form $\partial(\pi)$ for some partition π.
- Show that if $\mathcal{G}$ does not have two edge-disjoint spanning trees then player 1 can ensure player 2 does not win. The previous part might be helpful.

Exercise 2)

Marks: 3

Let $\mathcal{G} = (V, E)$ be an undirected graph. Let $\mathcal{I} = \{F \subseteq E : \text{each component of } (V; F) \text{ has at most one cycle}\}$. Show $\mathcal{M} = (E, \mathcal{I})$ is a matroid.

A direct proof is possible (though tedious). A more elegant proof shows that this supposed matroid is actually of a type of matroid we already saw in class. You may assume anything on the list of matroid examples from Lecture 26 is a matroid, even if we did not prove it.

Exercise 3)

Marks: 5

A common question asked in matroid theory is whether a matroid is *representable* over a field $\mathbb{F}$. That is, given a matroid $\mathcal{M} = (X, \mathcal{I})$ is there some matrix $\mathbf{A}$ over $\mathbb{F}$ whose columns are indexed by X such that $Y \in \mathcal{I}$ if and only if the columns of $\mathbf{A}$ indexed by Y are linearly independent?

- Let $\mathcal{M} = (E, \mathcal{I})$ be a graphic matroid over a graph $\mathcal{G} = (V; E)$. Pick an arbitrary direction for each edge $e \in E$. Let $\mathbf{A}$ be a matrix over $\mathbb{R}$ with rows indexed by V and columns indexed by E such that for each column $e = uv \in E$ we have $\mathbf{A}_{v,e} = 1$ and $\mathbf{A}_{u,e} = -1$, and $\mathbf{A}_{w,e} = 0$ for $w \notin \{u, v\}$. Show that $\mathcal{M} = (E, \mathcal{I})$ is the same as the vector matroid given by the columns of $\mathbf{A}$.

  Interesting Note: You can maybe see that $\mathbf{A}$ is totally unimodular. The class of matroids representable over $\mathbb{R}$ using totally unmodular matrices is, interestingly, the class of matroids representable over *all* fields.

- Let $\mathcal{H} = (L \cup R; E)$ be a bipartite graph. For each $e \in E$, let x_e be a variable over $\mathbb{Q}$ (the rational numbers). Let $\mathbf{A}$ be a matrix whose rows are indexed by L and whose columns are indexed by R where $\mathbf{A}_{uv} = x_{uv}$ if $uv \in E$ and $\mathbf{A}_{uv} = 0$ if $uv \notin E$.

  Consider any $U_L \subseteq L$ and $U_R \subseteq R$ with $|U_L| = |U_R|$. Let $\mathbf{A}'$ be the submatrix of $\mathbf{A}$ indexed by U_L and U_R. Show that $\det \mathbf{A}'$ is a nonzero polynomial if and only if the subgraph of $\mathcal{H}$ induced by $U_L \cup U_R$ has a perfect matching.

  Hint: Think of how permutations in the definition of determinant correspond to subsets of edges of such a matrix.

- For this exercise, you may assume (without proof), the following result.

  Lemma 1 *Let $f_1, f_2, \ldots, f_k \in \mathbb{Q}[x_1, x_2, \ldots, x_m]$ be nonzero polynomials. Then there are values $z_1, z_2, \ldots, z_m \in \mathbb{Q}$ such that $f_i(z_1, z_2, \ldots, z_m) \neq 0$ for each $1 \leq i \leq k$.*

  Using this and the previous part, show that every transversal matroid is representable by a matrix over $\mathbb{Q}$.

  Interesting Note: One can even show that every transversal matroid is representable over a sufficiently large finite field using essentially the same proof with the above lemma replaced by an application of the Schwartz-Zippel Lemma:
  `http://eccc.hpi-web.de/report/2010/096/`

Exercise 4)

Marks: 4

The image associated with this exercise has some colours that are crucial to the understanding of this problem. Let me know if you are having difficulty seeing the different colours in the picture.

Let $\mathcal{M}_1 = (E, \mathcal{I}_1)$ be the graphic matroid given by the graph below and let $\mathcal{M}_2 = (E, \mathcal{I}_2)$ be the following partition matroid over E. Each $e \in E$ is coloured either **red**, **green**, or **blue** as depicted in the figure. Then $F \subseteq E$ is in $\mathcal{I}_2$ if F contains at most 2 red edges, at most 1 green edge, and at most 3 blue edges.

Finally, let $F \subseteq E$ be the thick edges in the picture (the thinner dashed edges are $E - F$).

- Construct the graph $\mathcal{G}_F$, the bipartite directed graph we used in the matroid intersection algorithm.
- Find a shortest path from $F_1 = \{e \in E - F : F + e \in \mathcal{I}_1\}$ to $F_2 = \{e \in E - F : F + e \in \mathcal{I}_2\}$ and record the resulting set $F' \in \mathcal{I}_1 \cap \mathcal{I}_2$ obtained by alternating F along this path.
- Demonstrate some F_1 to F_2 path such that alternating F along this path does **not** produce a set $F' \in \mathcal{I}_1 \cap \mathcal{I}_2$. Thus, we see it is important to alternate along **shortest** paths.

  Note: Your example will have to use an item of F_1 or F_2 as an intermediate node on the path. It is possible to come up with an example where augmenting along one such path that excludes F_1 or F_2 as intermediate nodes also creates dependent sets.