CMPUT 675 - Assignment #0

Fall 2016, University of Alberta

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This assignment does not count toward your final grade. It is completely optional, you may choose to skip it without having your grade negatively affected. You can use this to get some experience with how I grade assignments. Feel free to only attempt some questions.

I will "grade" any submission within 3 days of receiving it as long as you submit it before the due date of Assignment #1. I offer no guarantees if you submit after.

Exercise 1)

Marks: 5

Let $\mathcal{G} = (V; E)$ be an undirected graph. For a vertex $v \in V$, let $\deg(v)$ be the number of edges $e \in E$ having v as an endpoint.

A **colouring** of G with k colours is an assignment $\phi : V \to \{1, \ldots, k\}$ such that $\phi(u) \neq \phi(v)$ for every edge $(u, v) \in E$.

Let $\Delta = \max_{v \in V} \deg(v)$. Describe a simple greedy algorithm to colour G with $\Delta + 1$ colours.

Exercise 2)

Marks: 5

All vectors in this exercise are column vectors. For two vectors $\mathbf{x}, \mathbf{y}$ of the same dimension, let $\mathbf{x} \leq \mathbf{y}$ mean $\mathbf{x}_i \leq \mathbf{y}_i$ for each index i. We let $\mathbf{A}^{\mathrm{T}}$ be the transpose of a matrix $\mathbf{A}$ and $\mathbf{0}_n$ be the n-dimensional vector consisting only of 0s.

Later in the course we will see the following.

Theorem 1 (Farkas' Lemma) *Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix and $\mathbf{b} \in \mathbb{R}^m$ be a vector. Then precisely one of the following is true:*

a) There is a vector $\mathbf{x} \in \mathbb{R}^n$ such that i) $\mathbf{x} \geq \mathbf{0}_n$ and ii) $\mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}$.

b) There is a vector $\mathbf{y} \in \mathbb{R}^m$ such that i) $\mathbf{y} \geq \mathbf{0}_m$, ii) $\mathbf{A}^{\mathrm{T}} \cdot \mathbf{y} = \mathbf{0}_n$ and iii) $\mathbf{b}^{\mathrm{T}} \cdot \mathbf{y} = -1$.

For now, show that a) and b) cannot hold at the same time.

Exercise 3)

Marks: 5

Let $\mathcal{G} = (V; E)$ be an undirected graph. A *matching* is a subset of edges $M \subseteq E$ such that no two edges in F share an endpoint. We will see in the course how to compute a maximum-size matching in polynomial time.

For now, consider the following simpler algorithm.

Algorithm 1 Greedy Matching

$M \leftarrow \emptyset$
while there is some edge $e \in E - M$ such that $M \cup \{e\}$ is a matching **do**
  $M \leftarrow M \cup \{e\}$
end while
return F

Show that the returned matching M satisfies $|M| \geq \nu(G)/2$ where $\nu(G)$ is the cardinality of a maximum-size matching in G.

Exercise 4)

Marks: 5

A connected component of a graph $\mathcal{G} = (V; E)$ is a nonempty subset of nodes $S \subseteq V$ where for every $u, v \in S$ there is a path from u to v in $\mathcal{G}$ and for every $u \in S, v \in V - S$ there is no path from u to v in $\mathcal{G}$. The connected components of $\mathcal{G}$ constitute a partition of V; for every $v \in V$ there is precisely one connected component S containing v.

Now let $E_1, E_2 \subseteq E$ be two subsets of edges and suppose the graph $\mathcal{F} = (V, E_1)$ has fewer connected components than $\mathcal{H} = (V, E_2)$. Show there is some edge $uv \in E_1$ such that u and v lie in different connected components in $\mathcal{H}$.