CMPUT 675 - Assignment #5

Fall 2014, University of Alberta
Due December 5 by 5:00 pm in my office.

Pages: 2

This assignment is to be completed individually. I understand that you may want to discuss the assignment with other students, a good guide for understanding my expectations is that you should not take notes or work out precise details in your discussions (keep it high-level). Mention any discussions and cite any resources you used on the writeup you hand in.

To be clear, whenever you are asked to give an approximation algorithm for a problem, it is expected that you will both describe the algorithm and prove the claimed approximation guarantee. If you can only think of an algorithm with a worse approximation guarantee than I am asking for, then describe it anyway. You may get partial marks (though, it cannot be entirely trivial). The same goes with lower bounds.

This assignment will be a bit shorter because of the project.

Problem 1)

Marks: 2

Complete the one missing argument from our spanning tree discussion. Prove that if $\mathcal{L}$ is a laminar collection of subsets of V such that $|S| \geq 2$ for each $S \in \mathcal{L}$ then $|\mathcal{L}| \leq |V| - 1$.

Problem 2)

Marks: 3

Consider the following LP relaxation for the TRAVELING SALESMAN problem in a metric (V, d).

$$\begin{array}{rrcll} \text{minimize}: & \displaystyle\sum_{e \in E} d(e) \cdot x_e & & & \\ \text{subject to}: & x(\delta(v)) & = & 2 & \text{for each } v \in V \\ & x(\delta(S)) & \geq & 2 & \text{for each } \emptyset \subsetneq S \subsetneq V \\ & x & \geq & 0 & \end{array} \qquad \textbf{(LP-TSP)}$$

1. Let T be a minimum spanning tree. Show that $d(T) \leq OPT_{\mathbf{LP-TSP}}$. [1 mark]
   Hint: It shouldn't take much work as long as you invoke the appropriate result from the lectures.
2. Let $D \subseteq V$ be the nodes that have odd degree in the minimum spanning tree T. A *D-join* is a subset of edges F such that D is the set of odd-degree nodes in the graph (V, F). It can be shown that the cheapest D-join has cost at most the optimum value of the following LP

(and that this LP has integral extreme points):

$$\begin{array}{rrcll} \text{minimize}: & \sum_{e \in E} d(e) \cdot y_e & & & \\ \text{subject to}: & y(\delta(S)) & \geq & 1 & \text{for each } S \subseteq V \text{ such that } |S \cap D| \text{ is odd} \\ & y & \geq & 0 & \end{array}$$

Use this fact to show that the cheapest D-join has cost at most $OPT_{\mathbf{LP-TSP}}/2$ (again, the optimum value of (**LP-TSP**). [1 mark]

3. Conclude that the integrality gap of (**LP-TSP**) is at most 3/2. [1 mark]

Problem 3)

Marks: 4 + 1 Bonus

- In the problem MAX-2LIN(3), we are given variables $x_1, \ldots, x_n$ over integers mod 2. Additionally, we are given constraints of the form $x_i + x_j + x_k \equiv b \pmod 2$ where $1 \leq i < j < k \leq n$ and $b \in \{0, 1\}$. The goal is to assign 0/1 values to the variables to maximize the number of satisfied constraints.

  It is known that unless $\mathbf{P} = \mathbf{NP}$, then for any constant $\epsilon > 0$ it is NP-hard to decide if at least a $(1 - \epsilon)$-fraction of constraints can be satisfied by some truth assignment or if at most a $(1/2 + \epsilon)$-fraction of the constraints are satisfied by any assignment.

  Use this (or any other fact) to show that MINIMUM VERTEX COVER cannot be approximated within some constant factor $\alpha > 1$ unless $\mathbf{P} = \mathbf{NP}$. Explicitly describe the value of α you find.

  Full marks will be awarded for showing a hardness of $\alpha = 7/6 - \epsilon$ for any constant $\epsilon > 0$, but partial marks will be assigned for any correct proof with some constant $\alpha > 1$. For example, a conceptually simpler hardness proof with a slightly smaller value α can be obtained from the $1 - \epsilon$ vs. $7/8 + \epsilon$ hardness for MAX-3SAT. [2 marks]

- Show that $\mathbf{PCP}_{1,c}(O(\log n), 2) = \mathbf{P}$ for any $c < 1$. [2 marks]
  Hint: The problem of deciding if all clauses in a 2SAT instance can be satisfied is in $\mathbf{P}$.

- **Bonus**: Let $f(n)$ be any function with growth rate $o(\log n)$ and let $q \geq 1$ be a constant. Show that if $\mathbf{PCP}_{1,1/2}(f(n), q) = \mathbf{NP}$ then in fact $\mathbf{P} = \mathbf{NP}$. [1 mark]