CMPUT 675 - Assignment #3

Fall 2014, University of Alberta
Due October 29 in class.

Pages: 4

This assignment is to be completed individually. I understand that you may want to discuss the assignment with other students, a good guide for understanding my expectations is that you should not take notes or work out precise details in your discussions (keep it high-level). Mention any discussions and cite any resources you used on the writeup you hand in.

To be clear, whenever you are asked to give an approximation algorithm for a problem, it is expected that you will both describe the algorithm and prove the claimed approximation guarantee. If you can only think of an algorithm with a worse approximation guarantee than I am asking for, then describe it anyway. You may get partial marks (though, it cannot be entirely trivial).

Problem 1)

Marks: 5

Consider the Max Dicut problem. We are given a directed graph $G = (V, E)$ with edge weights $w(u, v) \geq 0, (u, v) \in E$. The goal is to find a cut $S \subseteq V$ that maximizes the total weight of edges in $\delta^{out}(S) = \{(u, v) \in E : u \in S, v \notin S\}$.

- **Warmup**: Suppose we form S by placing each $v \in S$ with probability $1/2$. Show that the expected weight of all edges in $\delta(S)$ is exactly $\sum_{(u,v)\in E} w(u, v)/4$. [1 mark]
- Now consider the following LP-based approximation. For each directed edge $(u, v) \in E$ we have a variable $x_{u,v}$ and for each vertex $v \in V$ we have a variable y_v. The relaxation is the following:

$$\begin{array}{lrcll} \text{maximize :} & \displaystyle\sum_{(u,v)\in E} w(u,v) \cdot x_{u,v} & & & \\ \text{subject to :} & x_{u,v} & \leq & y_u & \text{for each } (u,v) \in E \\ & x_{u,v} & \leq & 1 - y_v & \text{for each } (u,v) \in E \\ & x_{u,v}, y_v & \in & [0,1] & \text{for each } (u,v) \in E \text{ and } v \in V \end{array} \qquad \textbf{(LP-Q1)}$$

  Argue that the optimum solution to (**LP-Q1**) is an upper bound on the optimum solution to the Max Dicut problem. [1 mark]

- Let $\mathbf{x}^*, \mathbf{y}^*$ be an optimum solution to (**LP-Q1**). Suppose we form S by placing each $v \in V$ in S with probability $1/4 + y^*_v/2$. Show $\Pr[(u, v) \in \delta^{out}(S)] \geq x^*_{u,v}/2$ for each $(u, v) \in E$. Conclude that this is a $1/2$-approximation. [2 marks]
- Show that this analysis is tight: for any $c > 1/2$ there is an instance of Max Dicut such that the relaxation (**LP-Q1**) for this instance has integrality gap $\leq c$. [1 mark]

Problem 2)

Marks: 2

A *star* is a tree where all but one vertex is a leaf. Show that if there is an α-approximation for the MULTICUT problem in stars then there is an α-approximation for the MINIMUM VERTEX COVER problem in general graphs. [2 marks]

Problem 3)

Marks: 4

Recall the LP relaxation for MINIMUM WEIGHT VERTEX COVER we saw in class. Let $G = (V, E)$ be an undirected graph with vertex weights $w(v), v \in V$.

$$\begin{array}{rrcll} \text{minimize :} & \sum_{v \in V} w(v) \cdot x_v & & & \\ \text{subject to :} & x_u + x_v & \geq & 1 & \text{for each } (u, v) \in E \\ & x_v & \in & [0, 1] & \text{for each } v \in V \end{array} \qquad \textbf{(LP-Q3)}$$

Let OPT_{LP} denote the optimum solution value to this LP for a given instance.

1. Prove that any extreme point solution $\bar{\mathbf{x}}$ is *half integral*: $\bar{x}_v \in \{0, 1/2, 1\}$ for each $v \in V$. **Hint**: Show that if $\bar{\mathbf{x}}$ is not half integral, then it can be expressed as a convex combination of two other LP solutions $\mathbf{x}', \mathbf{x}''$ where for every v with $\bar{x}_v \in \{0, 1/2, 1\}$ we have $\bar{x}_v = \bar{x}'_v = \bar{x}''_v$. [2 marks]

2. Show that if G can be coloured with χ colours, then the integrality gap is in fact at most $2 - 2/\chi$. Furthermore, show that if we are given an colouring of G with χ colours then we can find a vertex cover with total weight at most $2 - 2/\chi$ times the optimum solution value of (**LP-Q3**). [1 marks]

3. The previous part of this problem shows that the integrality gap of (**LP-Q3**) is at most $4/3$ when G is 3-colourable. Suppose we have a polynomial time algorithm $\mathcal{A}$ that finds a vertex cover of size at most $4/3$ times the optimum value of (**LP-Q3**) for any instance of MINIMUM VERTEX COVER (without vertex weights) where the graph G is 3-colourable, even if we do not know the colouring of G.

   Show that if we are given a graph G that we are told is 3-colourable without being given the colouring, then we can use $\mathcal{A}$ to efficiently colour G with $O(\log |V|)$ colours. [1 mark]

Problem 4)

Marks: 3

Consider the FACILITY LOCATION WITH PENALTIES problem. We have a set of client locations C and facility locations $F$with metric costs $c(i, j)$ between locations. Additionally, for each $i \in F$ we have an opening cost $f_i \geq 0$ and for each $j \in C$ we have a penalty $\pi_j \geq 0$. We must open some facilities and assign some clients to these open facilities while minimizing the overall cost. Here, the cost of a solution is the total facility opening cost, the total client assignment cost, and the total penalty for clients that are not assigned to any facility.

Now consider the following LP relaxation for the problem, where $x_{i,j} = 1$ indicates $j \in C$ gets assigned to $i \in F$, $y_i = 1$ indicates we open facility $i \in F$, and $z_j = 1$ indicates that we are choosing to not assign client j to any open facility.

$$\begin{array}{rrcll} \text{minimize :} & \multicolumn{4}{l}{\displaystyle\sum_{i \in F} f_i \cdot y_i + \sum_{i \in F, j \in C} c(i,j) \cdot x_{i,j} + \sum_{j \in C} \pi_j \cdot z_j} \\ \text{subject to :} & \displaystyle\sum_{i \in F} x_{i,j} + z_j & = & 1 & \text{for each } j \in C \\ & y_i - x_{i,j} & \geq & 0 & \text{for each } i \in F \text{ and each } j \in C \\ & \mathbf{x}, \mathbf{y}, \mathbf{z} & \geq & 0 & \end{array} \qquad \textbf{(LP-Q4)}$$

- Write the dual (**LP-Q4**). [1 mark]
- Let $\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*$ denote an optimal solution to (**LP-Q4**) with cost OPT_{LP}. Show

  $$\sum_{j \in C : z_j^* > 0} \pi_j \leq OPT_{\text{LP}}.$$

  Conclude the the integrality gap of this LP relaxation is at most $\alpha + 1$ where α denotes an upper bound on the integrality gap of the UNCAPACITATED FACILITY LOCATION LP relaxation from class. [2 marks]

Problem 5)

Marks: 2

Recall the CONSTRAINED FOREST problem we saw in class (the generalization of STEINER FOREST). We have an undirected graph $G = (V, E)$ with edge costs $c_e \geq 0, e \in E$. Furthermore, we also have a *cut requirement function* $f : 2^V \to \{0, 1\}$. Here, f is not explicitly given in the input; instead we have an efficient algorithm that takes a subset $S \subseteq V$ and outputs $f(S)$.

A subset $F \subseteq E$ is said to be feasible $\delta(S) \cap F \neq \emptyset$ for every $S \subseteq V$ with $f(S) = 1$. The goal is to find the cheapest feasible subset of edges. We saw that if f is a *proper* function then the integrality gap of a natural LP relaxation is at most 2.

Suppose instead that f is a *downward monotone* function:

- $f(\emptyset) = f(V) = 0$
- $f(S) \geq f(T)$ for every $\emptyset \subsetneq S \subseteq T$

For example, the function f where $f(S) = 1$ if and only if $1 \leq |S| \leq k - 1$ for some integer k is downward monotone. This models the problem of finding the cheapest subset of edges F so that every component in (V, F) has at least k vertices.

We can approximate the CONSTRAINED FOREST problem with a downward monotone cut requirement function in basically the same way as we approximated it for proper functions. Here, you are asked to provide the only significantly different detail.

Your Job
Suppose F is a feasible set of edges (i.e. $\delta(S) \cap F \neq \emptyset$ for any $S \subseteq V$ with $f(S) = 1$) such that $F - \{e\}$ is not feasible for any $e \in F$ (i.e. F is a minimial feasible set). Call a vertex $v \in V$ *active* if $f(\{v\}) = 1$ and *inactive* if $f(\{v\}) = 0$.

Show that in every connected component C of the graph $H = (V, F)$ there is at most one inactive vertex. Conclude that the average degree of the active vertices in H is at most 2. For partial marks, simply show this claim in the restricted setting where each $v \in V$ has some reward $r(v) \geq 0$ and $f(S) = 1$ if and only if $\sum_{v \in S} r(v) < k$ for some value k. [2 marks]

Note: This essentially shows the integrality gap of the LP relaxation we used for the CONSTRAINED FOREST problem is at most 2 when f is a proper function. The graph H above corresponds to the graph of contracted components in a particular dual growing iteration (so V here is really a collection of contracted sets of vertices in the original graph). The only other details that need to be modified from the proof for proper functions are very minor.