CMPUT 672 Notes on intersection graphs

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Let $\mathcal{S} = \{S_1, S_2, \ldots S_n\}$. The *intersection*, *overlap*, *containment*, and *disjointedness* graphs of $\mathcal{S}$ have:

vertices: $v_1, v_2, \ldots v_n$

edges: v_i and v_j are connected by an edge
iff $i \neq j$ and S_i and S_j satisfy:

intersection graph	$S_i \cap S_j \neq \emptyset$
overlap graph	$S_i \cap S_j \neq \emptyset$ and $S_i \not\subseteq S_j$ and $S_j \not\subseteq S_i$
containment graph	$S_i \subset S_j$ or $S_j \subset S_i$
disjointedness graph	$S_i \cap S_j = \emptyset$

$\mathcal{S} = \{S_1, S_2, \ldots S_n\}$ is an intersection (respectively, overlap, containment, or disjointedness) *model* for the graph. The *size* of the model is $|\cup_{i=1}^{n} S_i|$.

Every graph has: [Szpilrajn-Marczewski 45]
- an intersection model
- an overlap model
- a disjointedness model

Graph G has a containment model iff
G is a comparability graph.

Let $\mathcal{S} = \{S_1, S_2, \ldots S_n\}$.
If no containment then the intersection graph of $\mathcal{S} \equiv$ the overlap graph of $\mathcal{S}$.
If no two sets are disjoint then the overlap graph of $\mathcal{S} \equiv$ the *complement* of the containment graph of $\mathcal{S}$.

Given a graph:
what is the minimum size of a model?
can a (small) model be constructed efficiently?
is a model of at most a certain size guaranteed to exist?

For intersection models:

- intersection number (i.e. minimum size of an intersection model) = minimum size of an edge clique cover [Erdös, Goodman, Pósa 66]
- There exists a model $\mathcal{S}$ with $|\cup_{S \in \mathcal{S}} S| \leq n^2/4$ where n is the number of vertices of G.
  [Erdös, Goodman, Pósa 66]
- Determining whether or not there exists a model with $|\cup_{S \in \mathcal{S}} S| \leq k$ is NP-complete.
  [Kou, Stockmeyer, Wong 78]
- Determining whether or not there exists a model with $\max_{S \in \mathcal{S}} |S| \leq k$ is NP-complete.
  [Poljak, Rödl, Turzik 81]

Given a graph class
is there an upper bound on the minimum size of a model for any graph in the class?
can a model be constructed efficiently?

intersection number can be computed in polynomial time for:
- triangle-free graphs - a minimum edge clique cover consists of all edges of the graph
- chordal graphs [Raychaudhuri 88]
- $(\{a,b,c,d,e\},\{ab,bc,cd,da,ae,be,ce,de\})$-free comparability graphs [Raychaudhuri 91]
- octahedral graphs, i.e. $\overline{(n/2)K_2}$ [Rosgen 05]

overlap number can be computed in polynomial time for:
- trees [West et al: http://www.math.uiuc.edu/ west/pubs/overlapt.pdf]
- paths, cycles, caterpillars [Rosgen]
- cliques, complete k-partite graphs [Rosgen]

Containment number can be computed in polynomial time for:
- independent sets [Rosgen 05]
- complements of paths and caterpillars (trees?) [Rosgen 05]

Given a graph class, can it be characterized as the intersection, overlap, containment, or disjointedness graphs of collections of sets? [Scheinerman 85] [Golumbic and Scheinerman 89]
If so, can the corresponding sets be characterized? constructed efficiently?

intervals on a line
- interval graphs
- circle graphs
- permutation graphs
- comparability graphs of interval orders

arcs on a circle
- circular-arc graphs
- circular-arc overlap graphs
[Kashiwabara et al 91]
- circular-arc containment graphs
$\equiv$ circular permutation graphs
[Nirkhe 87] [Rotem & Urrutia 82]

subtrees in a tree
- chordal graphs
- subtree overlap graphs
- comparability graphs

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