A Translation of *Sur deux propriétés des classes d'ensembles* by Edward Szpilrajn-Marczewski*

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Translators' notes.

Edward Szpilrajn-Marczewski's *Sur deux propriétés des classes d'ensembles* is a text much referenced in intersection graph theory, since the construction of section 3 of this article shows that every graph is the intersection graph of some set of sets. The goal of this translation is to make accessible the original context of this result to those who have little or no knowledge of French. In any translation, a balance must be struck between faithfulness to the source text and readability in the target text. We have made an effort to maintain the character of the original article, generally opting not to make changes to its form or content. We have, however, changed some of the mathematical symbols to reflect current usage; specifically, where the original article contains the symbols $()$, $\{\}$, $\cdot$, and 0, we use $\{\}$, $\langle\rangle$, $\cap$, and $\emptyset$, respectively.

1. Problems and results. Let us consider the following properties of an arbitrary class of sets, $\boldsymbol{K}$:

Property (s). Every subclass of $\boldsymbol{K}$ of pairwise disjoint sets is at most countable.

Property (k). Every uncountable subclass of $\boldsymbol{K}$ contains an uncountable subclass of sets that have pairwise common points.

These properties arise for example in the study of Souslin's well-known problem and in other research in set theory, topology, etc.[1]

Obviously:

(i) Each class exhibiting property (k) exhibits property (s).

Furthermore, it follows easily from a relation defined by M. W. Sierpiński (see no. 4, below), that:

(ii) There exists a class of sets exhibiting property (s) but not property (k).

The goal of this note is to study properties (s) and (k) from the point of view of cartesian multiplication. For property (k), there is no difficulty (see 6(i) and 6(iii)); as to property (s), M. Sierpiński has recently demonstrated[2] that:

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*Fund. Math. **33** (1945), p. 303–307. Permission to translate and disseminate via www obtained from the Institute of Mathematics of the Polish Academy of Sciences, March 2008.

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[1]See for example B. Knaster, *Sur une propriété caractéristique de l'ensemble des nombres réels*, Recueil Math. Moscou (to appear), and my note *Séparabilité et multiplication cartésienne d'espaces topologiques*, Fund. Math. **34** (to appear) with the literature cited there.

[2]W. Sierpiński, *Sur un problème de la théorie générale des ensembles*, this volume, p. 299–302.

(iii) There exist two classes $\boldsymbol{P}$ and $\boldsymbol{Q}$ of sets, such that $\boldsymbol{P}$ and $\boldsymbol{Q}$ exhibit property (s) and their cartesian product $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$[3] does not.

I will demonstrate the following

Theorem. The class $\boldsymbol{P}$ of sets exhibits property (k) if and only if the cartesian product $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$ of the class $\boldsymbol{P}$ and every class $\boldsymbol{Q}$ that exhibits property (s) also exhibits property (s).

This theorem and proposition (ii) directly imply the result (iii) of M. Sierpiński.

However, it is to be noted that the analogous problems concerning topological spaces — more specifically, topological spaces in which the class of open sets exhibits property (k) or (s) — remain open.[4]

2. Properties (k), (s) and (t). It will be useful here to formulate another property of classes of sets, although this property will play only an auxiliary role:

Property (t). Every uncountable subclass of $\boldsymbol{K}$ contains among its elements two disjoint sets and two sets that have common points.

We now state without proof several propositions that are easy to verify, concerning (k), (s) and (t):

(i) Each of the properties (k) and (s) is hereditary (that is, exhibited by the subsets of sets that exhibit it).

(ii) If the relation of disjointness, considered in class $\boldsymbol{P}$ of sets, is isomorphic to the same relation considered in class $\boldsymbol{Q}$, and if class $\boldsymbol{P}$ exhibits property (k) or property (s), then class $\boldsymbol{Q}$ exhibits the same property, respectively.

(iii) Every uncountable class that has property (t) possesses property (s) without having property (k).

(iv) Every class that has property (s) but not property (k), contains an uncountable class exhibiting property (t).

(v) A class $\boldsymbol{K}$ has property (k) if and only if each sequence $\langle E_\xi \rangle$ of type Ω of nonempty sets belonging to $\boldsymbol{K}$ admits a subsequence $\langle E_{\alpha_\xi} \rangle$ of the same type, such that $E_{\alpha_\eta} \cap E_{\alpha_\zeta} \neq \emptyset$ for $\eta < \zeta < \Omega$.

3. Disjointness of sets as the most general symmetric relation. Consider the following theorem:

A relation ϱ is symmetric[5] *and irreflexive*[6] *if and only if there exists a class $\boldsymbol{K}$ of nonempty sets such that the relation of disjointness $(E_1 \cap E_2 = \emptyset)$, considered in $\boldsymbol{K}$, is isomorphic to ϱ.*

The sufficiency of this condition is obvious. In order to demonstrate its necessity, let us suppose that ϱ is a symmetric and irreflexive relation, defined on a set R, and let us designate, for each $p \in R$, by $N(p)$ the class of which the elements are: the set $\{p\}$ and each set $\{p, x\}$ such that $x \in R$ and not $p\varrho x$. Finally, let $\boldsymbol{K}$ be the family of classes $N(p)$ where $p \in R$. For $p \neq q$ belonging to R, we have $\{p\} \in N(p)$ and $\{p\} \notin N(q)$; consequently, $N(p) \neq N(q)$. So, letting each $p \in R$ correspond to $N(p)$, we obtain a bijection between R and $\boldsymbol{K}$. Then, for two elements of R, $p \neq q$, if $N(p) \cap N(q) \neq \emptyset$, the class $N(p) \cap N(q)$ contains but a single element, namely, the set $\{p, q\}$. It follows that the relations

$$p\varrho q \tag{1}$$

and

$$N(p) \cap N(q) = \emptyset \tag{2}$$

are equivalent.

[3] that is, the class of sets $A \times B$ where $A \in \boldsymbol{P}$ and $B \in \boldsymbol{Q}$, where the symbol $A \times B$ denotes the cartesian product of the sets A and B in the usual sense, cf. W. Sierpiński, 1. c., p. 299.

[4] Cf. my previously cited note, no. 3.3.

[5] that is, $p\varrho q$ implies $q\varrho p$.

[6] that is, we never have $p\varrho p$.

The theorem is thus proven.

It is easy to see that, in the statement of the theorem, the relation of disjointness can be replaced by that of the intersection of distinct sets ($E_1 \cap E_2 \neq \emptyset$; $E_1 \neq E_2$); see for example 5(i).

4. A class of sets having property (s) but not property (k). Let us designate by σ the following relation (defined by M. Sierpiński[7]) on the set Z of ordinal numbers at most countable: for $\langle x_\xi \rangle$ being a fixed sequence of type Ω composed of distinct real numbers, let $\alpha\sigma\beta$ when either $\alpha > \beta$ and $x_\alpha > x_\beta$, or $\alpha < \beta$ and $x_\alpha < x_\beta$.

It is easily shown that, in each uncountable set contained in Z, there exist ordinal numbers α, β and γ such that $\alpha\sigma\beta$ and not $\beta\sigma\gamma$.

As a result, by virtue of no. 3, *there exists a class of sets exhibiting property (t)*, and therefore — by virtue of 2 (iii) — *exhibiting property (s), but not property (k).*

5. Reciprocally isomorphic classes of sets. Two classes $\boldsymbol{P}$ and $\boldsymbol{Q}$ of nonempty sets are said to be *reciprocally isomorphic* when the relation of disjointness in $\boldsymbol{P}$ is isomorphic to the relation of intersection of distinct sets in $\boldsymbol{Q}$. In other words, the classes $\boldsymbol{P}$ and $\boldsymbol{Q}$ are reciprocally isomorphic when a bijection exists between them such that, for $A_1 \neq A_2$ belonging to $\boldsymbol{P}$ and for B_1 and B_2 belonging to $\boldsymbol{Q}$ and corresponding to A_1 and A_2, respectively, the relations $A_1 \cap A_2 = \emptyset$ and $B_1 \cap B_2 \neq \emptyset$ are equivalent.

(i) For each class of sets, there exists another, reciprocally isomorphic to it.

Since the relation ι of intersection (of distinct sets) is symmetric and irreflexive in a given class $\boldsymbol{P}$, there exists by virtue of no. 3 a class $\boldsymbol{Q}$ such that the relation of disjointness in $\boldsymbol{Q}$ is isomorphic to ι; consequently, the classes $\boldsymbol{P}$ and $\boldsymbol{Q}$ are reciprocally isomorphic.

We can see directly that

(ii) Every class that is reciprocally isomorphic to a class exhibiting property (t) also exhibits this property.

6. Cartesian multiplication. Assume that each of the classes $\boldsymbol{P}$ and $\boldsymbol{Q}$ in question contains at least one nonempty set.

(i) [and (ii)] If the class $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$ possesses property (k) [property (s)], the classes $\boldsymbol{P}$ and $\boldsymbol{Q}$ also possess it.

Let $\emptyset \neq E \in \boldsymbol{P}$. The class $\{E\} \dot{\times} \boldsymbol{Q}$ is contained in $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$ and consequently it possesses property (k) [property (s)] by virtue of 2(i). The relation of disjointness in $\boldsymbol{Q}$ and in $\{E\} \dot{\times} \boldsymbol{Q}$ being isomorphic, the class $\boldsymbol{Q}$ also exhibits this property by virtue of 2 (ii). Obviously, class $\boldsymbol{P}$ also posesses it.

(iii) If $\boldsymbol{P}$ and $\boldsymbol{Q}$ possess property (k), the class $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$ also posesses it.

Consider a sequence $\langle A_\xi \times B_\xi \rangle$ of type Ω of nonempty sets belonging to $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$. By 2(v), there exists a sequence $\langle A_{\beta_\xi} \rangle$ $(\xi < \Omega)$ such that $A_{\beta_\eta} \cap A_{\beta_\zeta} \neq \emptyset$ for $\eta < \zeta < \Omega$ and a sequence $\langle B_{\beta_{\gamma_\xi}} \rangle$ $(\xi < \Omega)$ such that $B_{\beta_{\gamma_\eta}} \cap B_{\beta_{\gamma_\zeta}} \neq \emptyset$ for $\eta < \zeta < \Omega$. Letting $\alpha_\xi = \beta_{\gamma_\xi}$, we obtain $A_{\alpha_\eta} \cap A_{\alpha_\zeta} \neq \emptyset$ and $B_{\alpha_\eta} \cap B_{\alpha_\zeta} \neq \emptyset$ for $\eta < \zeta < \Omega$. As a result,

$$(A_{\alpha_\eta} \times B_{\alpha_\eta}) \cap (A_{\alpha_\zeta} \times B_{\alpha_\zeta}) \neq \emptyset \quad \text{for } \eta < \zeta < \Omega;$$

therefore, it follows from 2(v) that the class $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$ exhibits property (k).

(iv) If the classes $\boldsymbol{P}$ and $\boldsymbol{Q}$ are uncountable and reciprocally isomorphic, the class $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$ does not exhibit property (s).

By ordering $\boldsymbol{P}$ and $\boldsymbol{Q}$ in sequences $\boldsymbol{P} = \langle A_\xi \rangle$ and $\boldsymbol{Q} = \langle B_\xi \rangle$ such that the relations $A_\eta \cap A_\zeta = \emptyset$ and $B_\eta \cap B_\zeta \neq \emptyset$ are equivalent, and letting $\boldsymbol{R} = \langle A_\xi \times B_\xi \rangle$, we obtain an uncountable class $\boldsymbol{R} \subset \boldsymbol{P} \dot{\times} \boldsymbol{Q}$ of pairwise disjoint sets.

7. Proof of the theorem. We now establish the theorem stated in no. 1.

[7] W. Sierpiński, *Sur un problème de la théorie des relations*, Annali R. Sc. Sup. Pisa **2** (1933), p. 285–288.

Necessity. It is to be shown that, $\boldsymbol{P}$ being a class with property (k) and $\boldsymbol{Q}$ a class with property (s), the class $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$ possesses property (s).[8] Let us suppose, to the contrary, that there exists a sequence $\langle A_\xi \times B_\xi \rangle$ of type Ω of disjoint nonempty sets belonging to $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$. By 2(v), there exists a sequence $\langle A_{\alpha_\xi} \rangle$ where $\xi < \Omega$, such that $A_{\alpha_\eta} \cap A_{\alpha_\zeta} \neq \emptyset$ for $\eta < \zeta < \Omega$. Therefore, it follows from the relation

$$(A_{\alpha_\eta} \times B_{\alpha_\eta}) \cap (A_{\alpha_\zeta} \times B_{\alpha_\zeta}) = \emptyset$$

that $B_{\alpha_\eta} \cap B_{\alpha_\zeta} = \emptyset$. Consequently, the sequence $\langle B_{\alpha_\xi} \rangle$ constitutes an uncountable class of disjoint nonempty sets belonging to $\boldsymbol{Q}$, which is incompatible with the hypothesis on the property (s) of $\boldsymbol{Q}$.

Sufficiency. For $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$ exhibiting property (s), it follows from 6(ii) that the class $\boldsymbol{P}$ also exhibits it. Consequently, it remains to be shown that, for each $\boldsymbol{P}$ having property (s), but not property (k), there exists a class $\boldsymbol{Q}$ such that $\boldsymbol{Q}$ has property (s) and the class $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$ does not.

By 2(iv), there exists a class $\boldsymbol{R} \subset \boldsymbol{P}$ that has property (t) and, by 5(i), there exists a class $\boldsymbol{Q}$ reciprocally isomorphic to $\boldsymbol{R}$, and thus also exhibiting this property by virtue of 5(ii). It follows from 6(iv) that $\boldsymbol{R} \dot{\times} \boldsymbol{Q}$ does not exhibit property (s); thus, by 2(i), the class $\boldsymbol{P} \dot{\times} \boldsymbol{Q}$ does not exhibit property (s) either.

Warsaw, January 1942.

[8] I owe this proposition to Messrs. Lance et Wichik.