Why not use Predicate Calculus?

Eg: Consider diagnosing toothache:

1. $\forall p$ $\texttt{Symptom}(p, \texttt{Toothache}) \Rightarrow \texttt{Disease}(p, \texttt{Cavity})$

   Wrong – other factors cause toothaches:

2. $\forall p$ $\texttt{Symptom}(p, \texttt{Toothache}) \Rightarrow \texttt{Dis}(p, \texttt{Cavity}) \vee$ $\texttt{Dis}(p, \texttt{GumDisease}) \vee \texttt{Dis}(p, \texttt{ImpactedWisdom}) \vee \ldots$

   Too many! Maybe diagnostic:

3. $\forall p$ $\texttt{Disease}(p, \texttt{Cavity}) \Rightarrow \texttt{Symptom}(p, \texttt{Toothache})$

   Wrong – many other factors (on lhs)!

- Difficulties of Building Exhaustive KB

  Laziness: Just too many rules and contingencies.

  Theoretical Ignorance: No complete theory for the domain.

  Practical Ignorance: Don't have all the (patient) information available.

- **Probabilities** provide way of summarizing uncertainty from $\left\{ \begin{array}{l} \text{laziness} \\ \text{ignorance} \end{array} \right\}$

Using Probability

- Not everyone with cavity has toothache

  $\neg\ [\forall p\ \texttt{Disease}(p, \texttt{Cavity})\ \Rightarrow\ \texttt{Symptom}(p, \texttt{Toothache})\ ]$

  but. . .

  perhaps 80% do.

- "80%" summarizes factors required for cavity to cause toothache

  + patient has cavity & toothache (but unrelated)

  Remaining 20% summarizes

  all other possible causes of toothache

Meaning: An individual with cavity either has toothache, or not.

In 80% of situations where x has `Cavity`
(ie, indistinguishable from this situation
based on current knowledge)
x has `toothache`

Terms from Probability Theory

Random Variable:

$\texttt{Weather} \in \{\texttt{Sunny, Rain, Cloudy, Snow}\}$

Domain: Possible values a random variable can take.

(. . . finite set, $\Re$, . . .)

Probability distribution: mapping from domain to values in $[0,1]$

$$P(\texttt{Weather}) = \langle 0.7, 0.2, 0.08, 0.02 \rangle$$

$$\text{means} \quad \left\{ \begin{array}{lcl} P(\texttt{Weather = Sunny}) & = & 0.7 \\ P(\texttt{Weather = Rain}) & = & 0.2 \\ P(\texttt{Weather = Cloudy}) & = & 0.08 \\ P(\texttt{Weather = Snow}) & = & 0.02 \end{array} \right\}$$

Event: Each assignment

(eg, `Weather = Rain`)

is "event"

General Events

Boolean Combinations: Can have

Conjunction, Disjunction, Negation

of events:

$P($ `Weather = Rain` $\wedge$ `Card = 2S` $)$

$P($ `Weather = Rain` $\vee$ `Card = 2S` $)$

$P(\neg$ `(Weather = Rain)` $)$

Atomic Event: "Complete specification"

Conjunction of assignments to EVERY variable

Joint Probability Distribution:

Probability of every possible atomic event

Toothache	Cavity	$P(\cdots)$
T	T	0.04
T	F	0.01
F	T	0.06
F	F	0.89

n binary variables: 2^n entries

($2^n - 1$ independent values, as sum $= 1$)

A huge table!

Joint Probability Distribution is Sufficient

t	c	$P(\texttt{Toothache} = t,\ \texttt{Cavity} = c)$
$+$	$+$	0.04
$+$	$-$	0.01
$-$	$+$	0.06
$-$	$-$	0.89

- $P(\texttt{Cavity} \vee \texttt{Toothache}) =$
  $P(\ \texttt{Cavity} \wedge\ \texttt{Toothache}) +$
  $P(\ \texttt{Cavity} \wedge \neg\texttt{Toothache}) +$
  $P(\neg\texttt{Cavity} \wedge\ \texttt{Toothache})$
  $= \quad 0.04 + 0.01 + 0.06 = 0.11$

- $P(\texttt{Toothache}) =$
  $P(\texttt{Toothache}\wedge\texttt{Cavity}) + P(\texttt{Toothache}\wedge\neg\texttt{Cavity})$
  $= \quad 0.04 + 0.01 = 0.05$

- Atomic Events are sufficient
  ... but very unnatural

- Why not "connections"?

Conditional Probability

Conditional Probability: $P(\, A \,|\, B \,) \; =$
Probability of event A,
given that event B has happened.

$P(\,\texttt{Cavity}\,|\,\texttt{Toothache}\,) = 0.8$

In gen'l:

$$P(\, A \,|\, B \,) \; = \; \frac{P(\, A \wedge B \,)}{P(\, B \,)}$$

$$P(\, A \wedge B \,) \; = \; P(\, A \,|\, B \,)\, P(\, B \,)$$

Unconditional (prior) Probability: Probability of event *before evidence is presented*

$P(\,\texttt{Cavity}\,) = 0.01$
$\equiv$ probability that someone (from this population) has a cavity is 1 in 100

Evidence: Percepts that affects degree of belief in event

Conditional (posterior) Probability: Probability of event *after evidence is presented*

N.b., posterior probability can be COMPLETELY different than prior probability!

Bayes' Rule and Its Use

Diagnosis typically involves computing

$$P(\,\texttt{Hypothesis}\,|\,\texttt{Symptoms}\,)$$

What is $P(\,\texttt{Meningitis}\,|\,\texttt{StiffNeck}\,)$?

$\equiv$ prob that patient A has meningitis,

given that A has stiff neck?

Typically have . . .

- Prior prob of meningitis $P(\,\texttt{M}\,) = \frac{1}{50{,}000}$
- Prior prob of having a stiff neck $P(\,\texttt{SN}\,) = \frac{1}{20}$
- Prob that meningitis causes a stiff neck $P(\,\texttt{SN}\,|\,\texttt{M}\,) = \frac{1}{2}$

Bayes' Rule: $P(\,B\,|\,A\,) \quad = \quad \dfrac{P(\,A\,|\,B\,)\,P(\,B\,)}{P(\,A\,)}$

$$P(\,Y\,|\,X, E\,) \quad = \quad \frac{P(\,X\,|\,Y{,}E\,)\,P(\,Y\,|\,E\,)}{P(\,X\,|\,E\,)}$$

Eg: $P(\,\texttt{M}\,|\,\texttt{SN}\,) = \frac{P(\,\texttt{SN}\,|\,\texttt{M}\,)P(\,\texttt{M}\,)}{P(\,\texttt{SN}\,)} = \frac{0.5 \times 0.00002}{0.05} = 0.0002$

Note: Only 1 in 5000 stiff necks have meningitis. . .
even though SN is major symptom of M. . .

Comments on Bayes Rule

- Don't need $P(\texttt{SN})$ if have $P(\texttt{SN}\,|\,\neg\texttt{M})$:

$$
\begin{aligned}
P(\texttt{SN}) &= P(\texttt{SN, M}) &+& \; P(\texttt{SN, }\neg\texttt{M}) \\
&= P(\texttt{SN}\,|\,\texttt{M})\,P(\texttt{M}) &+& \; P(\texttt{SN}\,|\,\neg\texttt{M})\,P(\neg\texttt{M}) \\
P(\neg\texttt{M}) &= 1 - P(\texttt{M})
\end{aligned}
$$

- Given "sore neck", want to compare

  prob of meningitis $P(\texttt{M}\,|\,\texttt{SN}) = \frac{P(\texttt{SN}\,|\,\texttt{M})\,P(\texttt{M})}{P(\texttt{SN})}$

  prob of whiplash $P(\texttt{W}\,|\,\texttt{SN}) = \frac{P(\texttt{SN}\,|\,\texttt{W})\,P(\texttt{W})}{P(\texttt{SN})}$

  To compute "relative likelihood", don't need $P(\texttt{SN})$:

$$
\frac{P(\texttt{M}\,|\,\texttt{SN})}{P(\texttt{W}\,|\,\texttt{SN})} = \frac{P(\texttt{SN}\,|\,\texttt{M})\,P(\texttt{M})}{P(\texttt{SN}\,|\,\texttt{W})\,P(\texttt{W})} = \ldots
$$

- $P(Y\,|\,X) \;=\; \alpha\,P(X\,|\,Y)\,P(Y)$

  where $\alpha = \frac{1}{P(X)}$ is independent of Y.

Combining Evidence

- What is prob of `Cavity`,

  given {`Toothache`, `Catch`}?

  $P(\texttt{Cav} \mid \texttt{Ta, Ct})$

- Bayesian Update:

$$
\begin{aligned}
P(\texttt{Cav} \mid \{\}) &= P(\texttt{Cav}) \\
P(\texttt{Cav} \mid \texttt{Ta}) &= P(\texttt{Cav} \mid \{\}) \frac{P(\texttt{Ta} \mid \texttt{Cav})}{P(\texttt{Ta})} \\
P(\texttt{Cav} \mid \texttt{Ta, Ct}) &= P(\texttt{Cav} \mid \texttt{Ta}) \frac{P(\texttt{Ct} \mid \texttt{Ta, Cavity})}{P(\texttt{Ct} \mid \texttt{Ta})}
\end{aligned}
$$

Each time new evidence is observed

`(Toothache; Catch; ...)`,

belief in unknown `(Cavity)`

is multiplied by factor

that depends on new evidence.

(Note: independent of order of observations)

Using Independence

Note: needs 3rd order information:

$$P(\texttt{Ct} \mid \texttt{Ta},\ \texttt{Cav})$$

Not always available...

- But sometimes, INDEPENDENCE!

$$P(\texttt{Ct} \mid \texttt{Ta},\ \texttt{Cav}) = P(\texttt{Ct} \mid \texttt{Cav})$$

(Prob of symptom2, given disease and symptom1

$\equiv$ Prob of symptom2, given disease)

If so...

$$P(\texttt{Cav} \mid \texttt{Ta},\ \texttt{Ct}) = P(\texttt{Cav}) \frac{P(\texttt{Ta} \mid \texttt{Cav})}{P(\texttt{Ta})} \frac{P(\texttt{Ct} \mid \texttt{Cav})}{P(\texttt{Ct} \mid \texttt{Ta})}$$

- **ASSUMPTION is NOT ALWAYS TRUE!**

But when it is, just need 2nd order statistics!

- Even better:
Denominator is $P(\texttt{Ta})\, P(\texttt{Ct} \mid \texttt{Ta}) = P(\texttt{Ta},\ \texttt{Ct})$

Independent of `Cavity`;
just normalizing term!

Probability Theory

Axioms:

$$0 \leq P(A) \leq 1$$
$$P(\texttt{True}) = 1, \quad P(\texttt{False}) = 0$$
$$P(A \vee B) = P(A) + P(B) - P(A \wedge B)$$
$$P(A) + P(\neg A) = 1$$

Not arbitrary: If Agent1 assign prob that violate axioms, then

$\exists$ betting strategy s.t. Agent1 guaranteed to lose \$

Def'n: Independence of Variables:

Events A and B are independent $\Leftrightarrow$

$$P(A \wedge B) = P(A)\,P(B)$$
$$P(A \mid B) = P(A)$$
$$P(A \vee B) = 1-(1-P(A))(1-P(B))$$

Variables independent

$\Leftrightarrow$ independent for all values

$$\forall a, b \; P(A=a,\; B=b) = P(A=a) \times P(B=b)$$

Source of Numbers

Requires numbers: $P(X)$, $P(X \mid Y)$

Where do they come from?

Experiments: Empirical, frequentist approach

Prior prob's from actuary tables,

Conditional probability (symptom, given disease)

sensitivity/specificity (lab results)

Objectivist: Probabilities express some real aspects of the universe.

Subjectivist: Characterizes an agent's beliefs.

- Reference class problem...

  What are "equivalent cases"?

Motivation for Belief Nets

Challenge: To decide on proper action

Which treatment, given symptoms?

Where to move?

Where to search for info?

. . .

Need to know dependencies in world

between symptom and disease

between symptom_1 and symptom_2

between disease_1 and disease_2

. . .

Q: Full joint?

A: Too big ($\geq 2^n$)

Too slow (inference requires adding 2^k . . .)

Better:

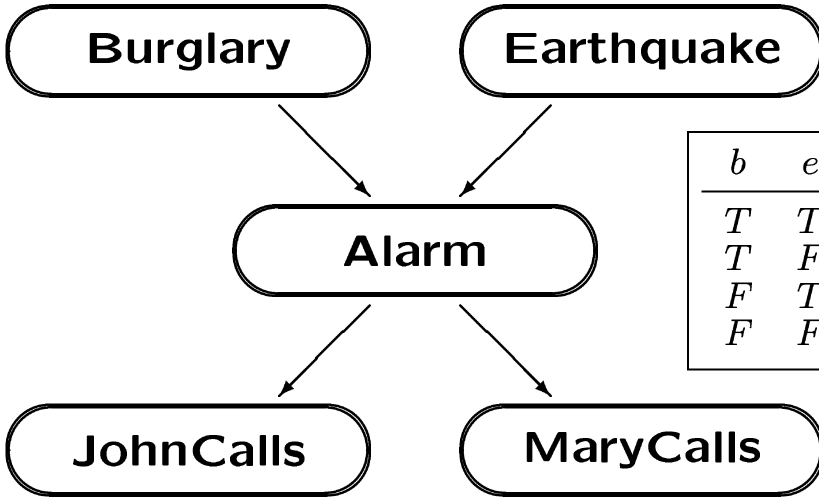
+ Encode dependencies

+ Encode *relevant* dependencies

Components of a Bayesian Net

$P(B)$
0.001

$P(E)$
0.002



b	e	$P(A \mid B=b, E=e)$
T	T	0.95
T	F	0.94
F	T	0.29
F	F	0.001

a	$P(J \mid A=a)$
T	0.90
F	0.05

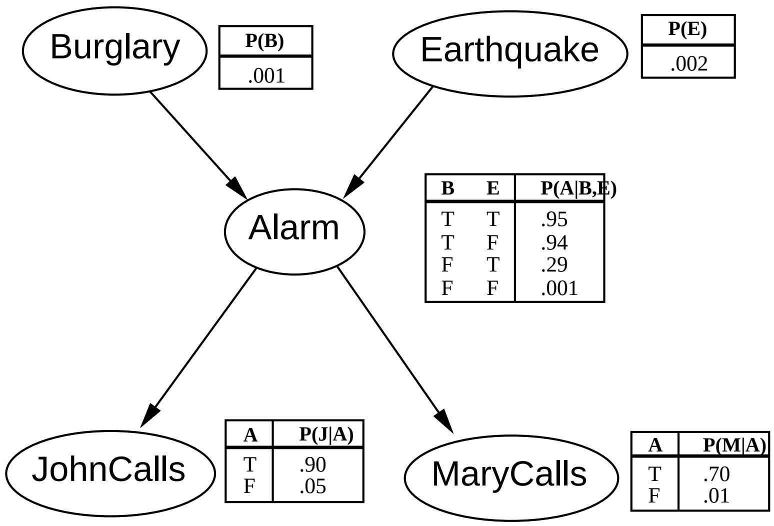
a	$P(M \mid A=a)$
T	0.70
F	0.01

Directed Acyclic Graph:

$$\mathcal{BN} = \left\{ \begin{array}{lll} \mathcal{N} & \text{Nodes} & \equiv \text{Variables} \\ \mathcal{A} & \text{Arcs} & \equiv \text{Dependencies} \\ \mathcal{C} & \text{CPTables} & \equiv \text{“weights”} \end{array} \right\}$$

- **Nodes:** one for each *random variable*
- **Arcs:** one for each *direct influence* between two random variables
- **CPT:** each node stores a conditional probability table
  $P(\texttt{Node} \mid \texttt{Parents(Node)})$
  to quantify effects of “parents” on child

Causes, and Bayesian Net



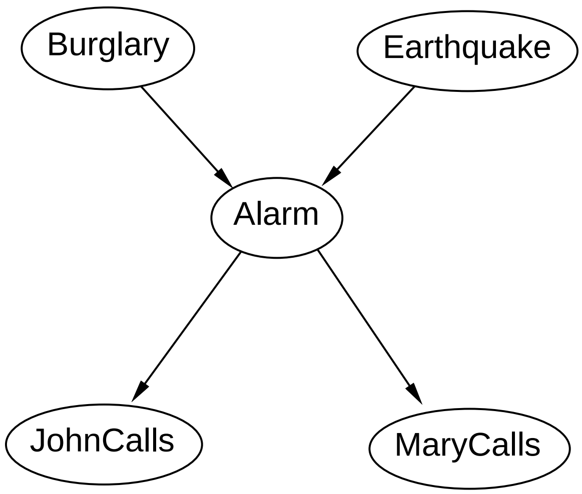
- What "causes" `Alarm`?
  A: `Burglary`, `Earthquake`
- What "causes" `JohnCall`?
  A: `Alarm`

  N.b., NOT Burglary, . . .
- Why not `Alarm` $\Rightarrow$ `MaryCalls`?

$$\left(\text{CPTable} = \boxed{\begin{array}{c|c} \text{Alarm} & P(\texttt{MC} \mid \texttt{A}) \\ \hline \text{T} & 1.0 \\ \text{F} & 0.0 \end{array}} \right)$$

  A: Mary not always home
  . . . may be playing loud music
  . . . phone may be broken
  . . .

Independence in a Bayesian Net



- `Burglary`, `Earthquake` independent
  (have no parents...)

- Given `Alarm`,
  `JohnCalls` and `MaryCalls` independent

`JohnCalls` is correlated with `MaryCalls` in general
as suggest `Alarm`

But given `Alarm`,

`JohnCalls` gives no NEW evidence wrt `MaryCalls`

Recovering Joint

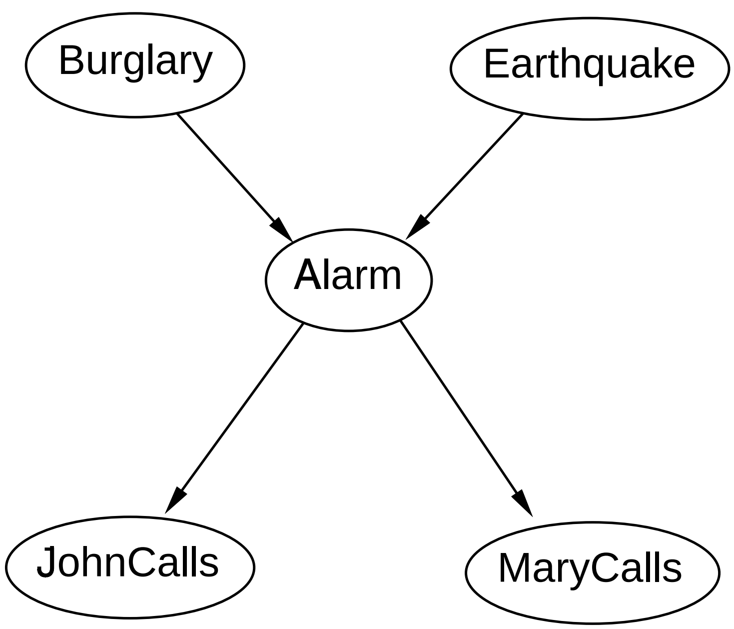
$$P(\neg b,\, e,\, a,\, \neg j,\, m) \quad =$$

$$P(\neg b)\; P(e \mid \neg b)\; P(a \mid e, \neg b)\; P(\neg j \mid a, e, \neg b)\; P(m \mid \neg j, a, e, \neg b)$$

$$P(\neg b)\; P(e)\; P(a \mid e, \neg b)\; P(\neg j \mid a)\; P(m \mid a)$$

$$0.99 \times 0.02 \times 0.29 \times 0.1 \times 0.70$$

Node independent of predecessors, given parents



$$P(\neg b,\, e,\, a,\, \neg j,\, m) \;=$$

$$P(\neg b) \cdot P(e) \cdot$$

$$P(a \mid \neg b, e) \cdot$$

$$P(\neg j \mid a) \cdot P(m \mid a)$$

Meaning of Bayesian Net

- A BN represents
  - + joint distribution
  - + condition independence statements

Eg: $P(J, M, A, \neg B, \neg E)$
$= P(J \mid A)\, P(M \mid A)\, P(A \mid \neg B, \neg E)\, P(\neg B)\, P(\neg E)$
$= 0.90 \times 0.70 \times 0.001 \times 0.999 \times 0.998$
$= 0.00062$

In gen'l, $P(X_1, X_2, \ldots, X_m) =$
$P(X_1 \mid X_2, \ldots, X_m)\, P(X_2, \ldots, X_m) =$
$P(X_1 \mid X_2, \ldots, X_m)\, P(X_2 \mid X_3, \ldots, X_m)\, P(X_3, \ldots, X_m) =$
$\prod_i P(X_i \mid X_{i+1}, \ldots, X_m)$

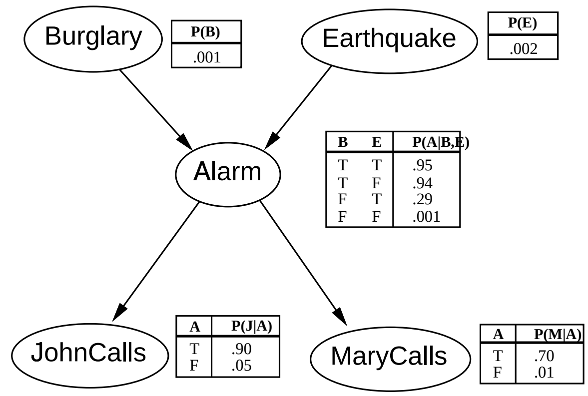
- Independence means:

$$P(X_i \mid X_{i+1}, \ldots, X_m) = P(X_i \mid Parents(X_i))$$

Node independent of predecessors, given parents

So. . . $$\boxed{P(X_1, X_2, \ldots, X_m) = \prod_i P(X_i \mid Parents(X_i))}$$

Comments



- BN used 10 entries
  ... can recover full joint (2^5 entries)

  (Given structure, other $2^5 - 10$ entries
  are REDUNDANT)

$\Rightarrow$ Can compute

$$P(\texttt{Burglary} \mid \texttt{JohnCalls}, \neg\texttt{MaryCalls}):$$

Get joint, then marginalize, conditionalize, ...

$\exists$ better ways...

Note: Given structure, ANY CPT is consistent.

$\nexists$ redundancies in BN...

Conditional Independence

Node X is independent of its non-descendants
given assignment to immediate parents $parents(X)$

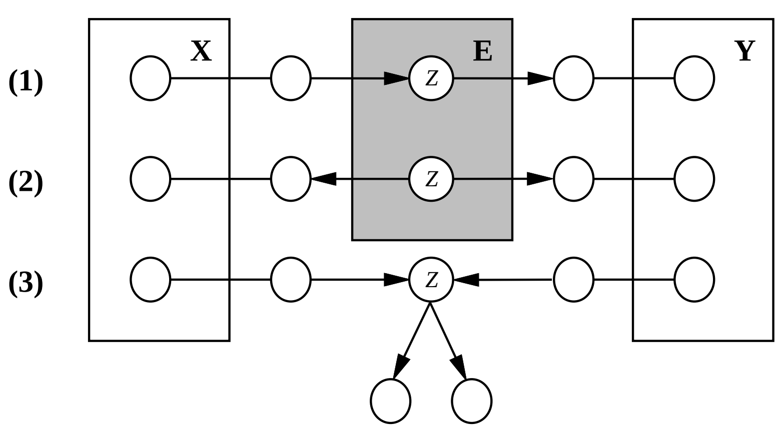
General question: "$X \perp Y \mid E$"
Are nodes X independent of nodes Y,
given assignments to (evidence) nodes E?

Answer: If every undirected path from X to Y
is d-separated by E, then $X \perp Y \mid E$

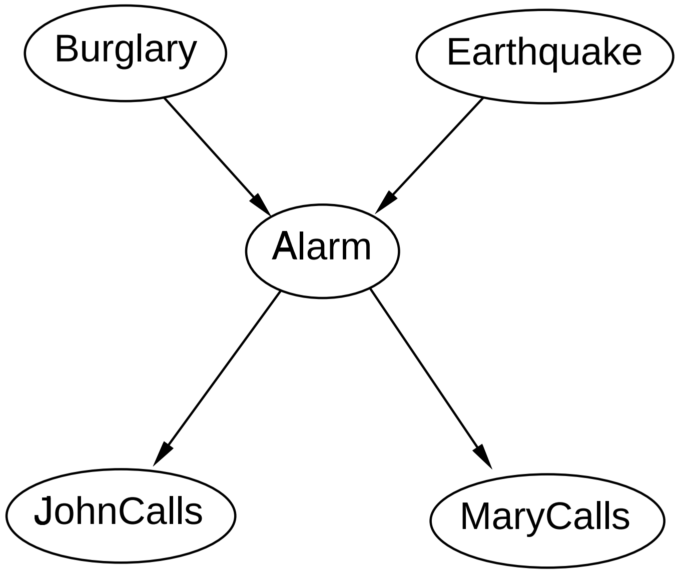
d-**separated** if every path from X to Y is **blocked** by E

... if $\exists$ node Z on path s.t.

1. $Z \in E$, and Z has 1 out-link (on path)
2. $Z \in E$, and Z has 2 out-link
3. Z has 2 in-links, $Z \notin E$, no child of Z in E



Explaining d-Separation



Case 1: `Burglary` and `JohnCalls` are *conditionally independent* given `Alarm`

Case 2: `JohnCalls` and `MaryCalls` are *conditionally independent* given `Alarm`

Case 3: `Burglary` and `Earthquake` are *independent* given no other information

But. . . `Burglary` and `Earthquake` are *dependent* given `Alarm`

Ie, `Earthquake` may "explain away" `Alarm` decreasing prob of `Burglary`