CMPUT 325 - Functional Programming

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14th September 2004

Functional Programming in *Lisp*

- The Theory of Functions

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- Example (polynomials)

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- E.g. Perhaps: G(rolling_stones,start_me_up) $\rightarrow$true
- Note each performer has many songs, each song can have multiple performers

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- Formally:
  - $f(d)$ defines **at least one** value: $\forall d \in D. \exists r \in R. f(d) = r$
  - $f(d)$ defines **at most one** value:

$$\forall d \in D. \forall r, s \in R. [f(d) = r \wedge f(d) = s] \Rightarrow r = s$$

Examples of Functions

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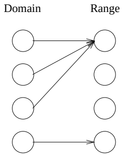
- The square of a number is a function.
  - Formally: $3^2 = 9$

Example Functions as Maps

Function	Domain Set	Range Set
age-of	all persons	positive reals
birth-mother	all people	people
square	numbers	numbers

Functions and Non-functions

Are these examples functions?

- 

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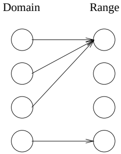

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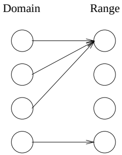
- 

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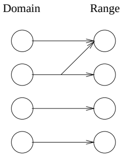
- 

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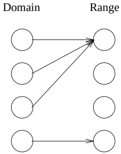
- 

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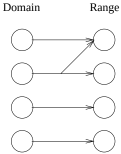
- 

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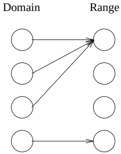
- 

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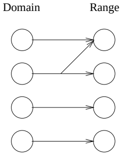
- 

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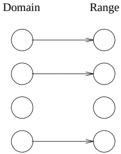
- 

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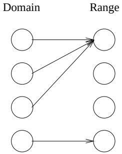
- 

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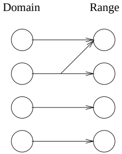
- 

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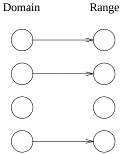
- 

No. Exists d with no r (e.g. `sqrt(d)`)

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- Sum is a binary function mapping tuples $(z_1, z_2) \in \mathcal{Z}^2$ to elements in $\mathcal{Z}$
  $$+ : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathcal{Z}$$

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- The rectangular-to-polar-coordinates function, $\mathrm{aspolar}$, maps $\mathcal{R}^2$ to radius-angle space $\mathcal{R} \times \Theta$

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- **NOTE: for each element of the domain a vector function returns a exactly one tuple**

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- Sample Range value: f(x) = if x>0 return 1 else return -1

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  - Postfix notation: function name follows arguments
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- Def'n: preimage or inverse of $r \in R$ under function f is the element(s) of $d \in D$ that result in r

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- Example: $! \circ +(2,3) \ = ![+(2,3)] \ = ![5] \ = \ 120$ $! \circ +$ is a function
  taking two real numbers, and
  returning factorial if their sum $\in \mathcal{Z}^+$