Reversible Moves and Combinatorial Decompositions in Hex

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Combinatorial Games

\[
0 = \{ \ | \ } \quad -1/2 = \{ -1 | 0 \} \quad 2* = \{ 2 | 2 \}
\]

\[
5 = \{ 4 | \} \quad * = \{ 0 | 0 \} \quad \uparrow = \{ 0 | * \}
\]

- Games have options for players Left and Right
- Games define a number system, and all numbers are games
- Games can be negated: reverse roles of players
- Games can be compared: Left prefers positive numbers, Right negative, etc
Reversible Moves

Player $P$’s move from game position $G_0$ to $G_1$ is reversible if $\overline{P}$ has a response to $G_2$ such that $G_0 \geq_P G_2$.
Reversible moves can be bypassed: replacing $P$'s reversible move option in $G_0$ with $P$'s options in $G_2$. 

\[
G_0 = \{G_1, X, Y, Z, \ldots | \ldots \} \\
G_1 = \{\ldots | G_2, \ldots \} \\
G_2 = \{A, B, C, \ldots | \ldots \} \\
G_0 = \{A, B, C, \ldots, X, Y, Z, \ldots | \ldots \}
\]
Vulnerable Moves are Dead-Reversible

- A vulnerable move is a play that can be rendered dead (useless) by an opponent response.
- This exchange essentially gives the opponent a free move.
- Extra stones can never be a disadvantage, so vulnerable moves are reversible.
The same thing is true of captured cells!
If player $\overline{P}$ has a move that causes cells $C$ to be $P$-captured, then $P$ should not consider any move in set $C$
Captured-reversible moves can be pruned, instead of only bypassed
Opposite colour chains are *touching* if there exists a pair of their stones that are adjacent or form an opposite-colour bridge.

A 4-cycle of touching chains forms a four-sided decomposition.
Four-sided Decompositions

- A four-sided decomposition is equivalent to a Hex game: no draws, try to connect opposing edges
- Only 3 possibilities: Black has a virtual connection, White has a virtual connection, or both players have semi-connections
- In the first two cases, fill-in is possible... what about the third case?
Only a unique move should be considered for each player: these moves will fill-in the four-sided decomposition for the player who moves first.

Similar to the star game in combinatorial game theory: *star decomposition*
If a star decomposition can be formed in multiple ways, which way is preferable?

If two potential star decompositions $S_1, S_2$ exist, with $S_1 \subseteq S_2$, then should always pick $S_2$
Star Decompositions (3)

- If claim first move in star decomposition, then claim larger region
- If opponent claims first move, they get same benefit as before, and you can claim the set difference $S_2 \setminus S_1$
Star Decompositions (4)
Strategy Decompositions

- Four-sided and star decompositions defined by regions of the board
- Value of each region can be analyzed independently, and then recombined to solve the whole game
- Strategic decompositions are not in separate regions, but are conceptual divisions: miai in Go, virtual and semi connections in Hex, etc
- Can we dynamically identify such decompositions during solver’s search?
If there exists a bridge, with one carrier cell vulnerable to the other, then only a single probe is possible.

If responding to the probe with the vulnerable cell is a winning move, then clearly both probes are losing.

Stronger result: then the probing player has no winning move!
Sketch of Proof:
- Adopt pairing strategy on bridge cells, and winning strategy for continuation of game after probe/defense exchange
- During game, will continue on rest of board as if opponent has claimed probe
- This assumption cannot be bad, as opponent can do no better on the bridge region
This proof does not require the pair to be a bridge

So if a $P$-move $M$ loses to $\overline{P}$-response $R$, where $R$ is $P$-vulnerable to $M$, then the original position is a win for $\overline{P}$.
Summary

- Identified more reversible moves in Hex
- Proved can prune (not just bypass) captured-reversible moves
- Found domination relationship among moves that form star decompositions
- Method for dynamically identifying strategy decompositions using properties of vulnerable cells
- Still need to incorporate these ideas into our Hex solver and players