A Handicap Hex Strategy

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Hex Basics

Rules

- Two players alternate turns playing on any empty cell
- Stones are permanent (no moving or capturing)
- Goal is to connect your two sides of the board
Hex Basics

Theoretical Properties

- An extra stone of your color is never a disadvantage
- Hex cannot end in a draw
- First-player win: strategy-stealing argument
- PSPACE-complete to determine winner in arbitrary position
Know only that there exists a winning first-player strategy
How many stones do we need to place initially to guarantee a win, and where should these handicap stones be placed?
Claude Berge would give three stone handicaps on the $11 \times 11$ board
On irregular Hex boards, player traversing shorter distance can win even as second-player using a simple pairing strategy.

Idea: use handicap stones to essentially reduce a regular Hex board to an irregular one.
Using graph-theoretic properties, can determine stones that can be added to Hex positions without changing its value.

Two types of fill-in: dead cells and captured cells.
Dead Cells

- A cell is dead if it is not on any minimal winning path (for either player).
- Dead cells can be filled-in with stones of either colour.
A set of cells $S$ is $P$-captured if player $P$ has a second-player strategy to make all $\overline{P}$-claimed $S$-cells dead.

$P$-captured sets of cells can be filled-in with $P$-coloured stones.
New: permanently inferior cells

Adds a single stone for one particular player $P$

Unlike captured, the strategy set is larger than the filled-in set
Permanently Inferior Strategy (1)

- If $\bar{P}$ moves first in pattern, claim $\bar{P}$ must play at shaded cell
- Any other move allows $P$ to capture all four cells
Permanently Inferior Strategy (2)

- If $\overline{P}$ plays at shaded cell, dotted cell is dead
- If $P$ plays first, captures all four cells
- In all cases, $P$ can claim the dotted cell
- Can assign $P$ the dotted cell without changing position’s value
Permanently Inferior Cell Patterns

- All three permanently inferior patterns
Handicap Stone Placement

- Placement of stones begins on second row, main diagonal.
- Stones placed every 6 spaces until last stone is at most distance four from the edge.
- On an $n \times n$ Hex board, this requires $\left\lceil \frac{n+1}{6} \right\rceil$ stones.
Capture cells below handicap stones
Handicap Reduction (2)

Apply permanently inferior cell patterns next
Fill-in remaining gaps via dead and/or captured
Essentially reduced to $n - 1 \times n$ board, so proven existence of winning strategy

Can easily make strategy explicit by enforcing inferior cell strategies
Summary

- New: permanently inferior cells
- Developed efficient and explicit handicap strategy for Hex

Open Questions:
- Are there more permanently inferior cell patterns?
- Can the number of handicap stones be further reduced while maintaining an explicit strategy?
- Can the number of handicap stones be reduced if we only desire an existence proof (while still specifying initial stone placement)?
Any Questions?

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