\[ E(f(x)) = \int_a^b F\left( f(x), \frac{df}{dx}, x \right) \, dx \]

Let's denote \( f = \frac{df}{dx} \).

Let's perturb \( f(x) \) by a small change \( f(x) + \varepsilon g(x) \) when \( \varepsilon > 0 \) is a small number and \( g(x) \) is another function.

So, change in \( E \) would be

\[
E\left( f(x) + \varepsilon g(x) \right) - E(f(x))
\]

\[
= \int_a^b \left[ F\left( f + \varepsilon g, \frac{df}{dx} + \varepsilon \frac{dg}{dx}, x \right) - F\left( f, \frac{df}{dx}, x \right) \right] \, dx
\]

\[
\approx \int_a^b \left[ \frac{\partial F}{\partial f} (\varepsilon g) + \frac{\partial F}{\partial \frac{df}{dx}} (\varepsilon \frac{dg}{dx}) \right] \, dx \quad \left[ \text{After Taylor series approximation} \right]
\]

\[
= \int_a^b \frac{\partial F}{\partial f} g \, dx + \varepsilon \int_a^b \frac{\partial F}{\partial \frac{df}{dx}} \frac{dg}{dx} \, dx - \varepsilon \int_a^b \frac{d}{dx}\left( \frac{\partial F}{\partial \frac{df}{dx}} \right) g \, dx
\]

\[
\left[ \text{Applying Integration by parts to the second term} \right]
\]
\[ \begin{align*}
\int_{a}^{b} \left[ \frac{\partial F}{\partial t} - \frac{d}{dx} \left( \frac{\partial F}{\partial x} \right) \right] g \, dx
\end{align*} \]

After applying boundary conditions, we can apply two types of boundary conditions to make this happen.

1. \( g(a) = g(b) = 0 \).

Or, (2) \( g(a) = g(b) \) and \( \frac{\partial F}{\partial x}(a) = \frac{\partial F}{\partial x}(b) \).

The first type arises when any two end points of \( f(x) \) are always fixed.

The second type typically arises when any \( f(x) \) is a closed curve (not necessarily to be vector valued), like in a closed shape or active contour.

If we choose \( g(x) = - \left[ \frac{\partial F}{\partial t} - \frac{d}{dx} \left( \frac{\partial F}{\partial x} \right) \right] \),

then \( E(f(x) + \varepsilon \delta(x)) - E(f(x)) = -3 \int \left[ \frac{\partial F}{\partial t} - \frac{d}{dx} \left( \frac{\partial F}{\partial x} \right) \right]^2 \, dx \leq 0 \).

So it is guaranteed that by this choice of \( g(x) \) the energy will never increase.
Also note that
\[ E(f(x + \varepsilon y)) - E(f(x)) = 0 \]
\[ \varepsilon \text{ if and only if } \frac{\partial F}{\partial f} - \frac{d}{dx} \left( \frac{\partial F}{\partial f} \right) = 0 . \]
So starting from initial few one can change the few by the following rule to decrease the energy \( E \).
\[ f(x) \leftarrow f(x) - \varepsilon \left[ \frac{\partial F}{\partial f} - \frac{d}{dx} \left( \frac{\partial F}{\partial f} \right) \right] \]
\[ \therefore \quad \frac{df}{dt} = -\left[ \frac{\partial F}{\partial f} - \frac{d}{dx} \left( \frac{\partial F}{\partial f} \right) \right] \quad \text{This eqn is called gradient descent equation.} \]

Steady state solution of it would be when there is no change of energy, i.e., when
\[ \frac{\partial F}{\partial f} - \frac{d}{dx} \left( \frac{\partial F}{\partial f} \right) = 0 \]
This eqn is called Euler eqn.
Let us apply this to a toy example. Finding shortest path between two points in a plane. The energy functional in this case is \[ E(t) = \int_a^b \sqrt{1 + (y')^2} \, dx \]
Euler eqn. in this case is

\[
\frac{\partial}{\partial t} \left[ \sqrt{1 + \frac{1}{\dot{t}^2}} \right] - \frac{d}{dx} \left[ \frac{3}{2} \sqrt{1 + \dot{t}^2} \right] = 0
\]

i.e.

\[
\frac{d}{dx} \left[ \frac{\dot{t}}{\sqrt{1 + \dot{t}^2}} \right] = 0 \implies \frac{d\dot{t}}{dx} = \frac{d^2\dot{t}}{dx^2}
\]

or,

\[
\frac{\ddot{t}}{\left[ 1 + \dot{t}^2 \right]^{3/2}} = 0 \quad \left[ \ddot{t} = \frac{d^2 t}{dx^2} \right]
\]

or,

\[
\frac{d^2 \dot{t}}{dx^2} = 0 \implies f(t) = Ax + B.
\]

a straight line, as we already know.

- If \( E = \int F(t, \dot{t}, \ddot{t}, x) \, dx \), then

\[
\frac{\partial E}{\partial t} = - \left[ \frac{\partial F}{\partial t} - \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{t}} \right) + \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial \ddot{t}} \right) \right]
\]

- If \( E = \int F(t, g, \dot{t}, \dot{g}, \ddot{t}, \ddot{g}, x) \, dx \), then there will be two gradient descent (or Euler) eqns:

\[
\frac{\partial E}{\partial t} = - \left[ \frac{\partial F}{\partial t} - \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{t}} \right) + \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial \ddot{t}} \right) \right]
\]

\[
\frac{\partial E}{\partial \dot{t}} = - \left[ \frac{\partial F}{\partial \dot{t}} - \frac{d}{dx} \left( \frac{\partial F}{\partial \dot{g}} \right) + \frac{d^2}{dx^2} \left( \frac{\partial F}{\partial \ddot{g}} \right) \right]
\]
Snake energy functional
\[ E(x, y) = \int \left[ \frac{1}{2} (\dot{x}^2 + \dot{y}^2) + \frac{\beta}{2} (\ddot{x}^2 + \ddot{y}^2) \right] ds \]

From the preceding rule, there will be two Euler eqns.
\[ \frac{\partial x}{\partial t} = - \left[ -q_x(x, y) - \alpha \frac{d}{ds} (\dot{x}) + \beta \frac{d^2}{ds^2} (\ddot{x}) \right] \]
\[ \frac{\partial y}{\partial t} = - \left[ -q_y(x, y) - \alpha \frac{d}{ds} (\dot{y}) + \beta \frac{d^2}{ds^2} (\ddot{y}) \right] \]

\[ \frac{d\mathbf{X}}{dt} = [q_x(x, y) + \alpha \frac{d^2}{ds^2} (\ddot{x}) - \beta \frac{d^2}{ds^2} (\ddot{y})] \]

Now, we need to discretize these two PDE's.

Let's have \( N \) points on the contour in total. Here, for the \( x \)-eqn,
\[ x_{n+1}^{t+1} - x_n^t \]
\[ = \frac{\beta}{\Delta t} \left( x_{n+1}^t - 4 x_n^t + 6 x_{n-2}^t - 4 x_{n+2}^t + x_{n+4}^t \right) \]
\[ x_n^t = x_n^t + (\Delta t) q_x(x_n^t, y_n^t) + \Delta t \left[ (\beta x_n^{t+1} + (\alpha + \beta) x_{n+1}^t - (2 + \alpha + \beta) x_n^t \right] \]

\[ x_n^{t+1} = x_n^t + (\Delta t) q_x(x_n^t, y_n^t) + (\alpha + \beta) \left[ (\beta x_n^{t+1} - (\alpha + \beta) x_{n+1}^t + (2 + \alpha + \beta) x_n^t - (\alpha + \beta) x_n^{t+1} \right] \]

This one is known as explicit scheme.
Explicit scheme has some problems in terms of stability. Unless the \((\alpha t)\) value is very small, the solution will diverge. So a better method is semi-implicit method:

\[
X_{n+1}^t = X_n^t + (\alpha t) \frac{\partial}{\partial x}(X_n^t, Y_n^t) - (\alpha t) \left[ \beta X_{n-1}^t - (\alpha + 4\beta) X_{n-1}^t + (2\alpha + 6\beta) X_{n+1}^t - (\alpha + 4\beta) X_{n+1}^t + \beta X_{n+2}^t \right]
\]

In matrix-vector form this can be written as:

\[
A \bar{X}^{t+1} = \bar{X}^t + (\alpha t) \bar{g}^t
\]

Where \(\bar{X}^t = [X_1^t, \ldots, X_N^t]^T\), \(\bar{X}^{t+1} = [X_1^{t+1}, \ldots, X_N^{t+1}]^T\)

\(\bar{g}^t = \left[ g_x(X_1,Y_1), \ldots, g_x(X_N,Y_N) \right]^T\)

So that

\[
\bar{X}^{t+1} = A^{-1} (\bar{X}^t + (\alpha t) \bar{g}^t)
\]

\[
A = \begin{bmatrix}
  c & b & a & \cdots & a \\
  b & c & b & a & \cdots \\
  a & b & c & b & \cdots \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a & b & c & b & a
\end{bmatrix}
\]

\[
c = 1 + (\alpha t)(2\alpha + 6\beta)
\]

\[
b = - (\alpha t)(\alpha + 4\beta)
\]

\[
a = (\alpha + 4\beta)
\]

Similarly, the iteration rule is:

\[
\bar{Y}^{t+1} = A^{-1} (\bar{Y}^t + (\alpha t) \bar{g}^t)
\]
Gradient Vector Flow.

Snake Eqn.

\[
\frac{\partial x}{\partial t} = \alpha \frac{d^2x}{ds^2} - \beta \frac{d^4x}{ds^4} + g_x(x,y)
\]

\[
\frac{\partial y}{\partial t} = \alpha \frac{d^2y}{ds^2} - \beta \frac{d^4y}{ds^4} + g_y(x,y)
\]

We can interpret these eqns. as a force balance eqn. when \( \alpha, \beta \) terms are internal force trying to balance the external force \( (g_x, g_y) \). So why not make use of this interpretation by designing a better external force \( u, v \) from the image data.

\[
E_{\text{envf}}(u, v) = \iint \left\{ \mu (u_x^2 + u_y^2 + v_x^2 + v_y^2) + \right. \\
\left. |\nabla|^2 \left[ (u-t) \right] + \left[ (v-f) \right] \right\} dx dy
\]

minimize \( E_{\text{envf}} \) to obtain \( (u, v) \) - the external force for snake.
Note that $E_{\text{avr}}$ is of the following form:

$$E_{\text{avr}} = \iint F(u, u_x, u_y, v, v_x, v_y, x, y) \, dx \, dy$$

For this form of the energy functional, the gradient descent eqns are:

$$\frac{\partial u}{\partial t} = - \left[ \frac{\partial F}{\partial u} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial u_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial u_y} \right) \right]$$

and

$$\frac{\partial v}{\partial t} = - \left[ \frac{\partial F}{\partial v} - \frac{\partial}{\partial x} \left( \frac{\partial F}{\partial v_x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial F}{\partial v_y} \right) \right]$$

So the PDE's for $\text{avr}$ become:

$$\frac{\partial u}{\partial t} = 2\mu \nabla^2 u - 2 |\nabla H|^2 (u - fx)$$

$$\frac{\partial v}{\partial t} = 2\mu \nabla^2 v - 2 |\nabla H|^2 (v - fy)$$

When $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, this operator is called the Laplacian operator.